On the classification of group actions on C*-algebras up to equivariant KK-equivalence

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Outline

Motivation

- The Elliott Classification Programme classifies simple, nuclear C^*-algebras with some extra properties up to isomorphism.
- This is almost finished.
- Purely infinite C^*-algebras are easier.
- Even non-simple purely infinite C^*-algebras are tractable:

Theorem (Kirchberg)

Let X be a topological space. Let A, B be strongly purely infinite, stable, nuclear C^*-algebras with primitive ideal spaces homeomorphic to X. Then A and B are isomorphic over X $\iff$ A and B are equivalent in $\mathrm{KK}(X)$.

How about classifying group actions on Kirchberg algebras?

- results for groups like $\mathbb{Z}$, $\mathbb{Z}^2$, poly-cyclic groups or actions with special properties
- I suggest a two-step process:

algebraic topology	classify objects in the bootstrap class in KK^G up to KK^G-equivalence
analysis	improve KK^G-equivalence to conjugacy under suitable assumptions

- My talk will only be about the algebraic topology aspects.
- The analysis aspect is not done.
- There are some encouraging results by Gábor Szabó.

Two opposite cases

- Direct proofs of classification theorems need groups without torsion or actions with the "Rokhlin property".
- The Rokhlin property limits the K-theory of A and $A \rtimes G$.
- We will study two opposite cases:
  1. torsion-free, amenable groups:
     {actions in the equivariant bootstrap class}/KK^G-equivalence correspond to principal bundles over BG
  2. groups of prime order, $\mathbb{Z}/p$ for a prime p:
     {actions in the equivariant bootstrap class}/KK^G-equivalence correspond to exact modules over a certain $\mathbb{Z}/2$-graded ring
- The modules in case 2 are complicated.
- As a test case, we completely describe actions on Cuntz algebras.

The equivariant bootstrap class

Definition (equivariant bootstrap class $\mathfrak{B}^G$)

localising subcategory of KK^G generated by $\mathrm{Ind}_H^G(\mathbb{M}_n(\mathbb{C}))$
for $H \subseteq G$ compact and $H \curvearrowright \mathbb{M}_n(\mathbb{C})$,
coming from a projective representation $H \curvearrowright \mathbb{C}^n$

Theorem (Dell'Ambrogio–Emerson–Meyer 2014)

An object of KK^G belongs to $\mathfrak{B}^G$ if and only if it is KK^G-equivalent to an action on a Type I C^*-algebra.

Remark

$\mathfrak{B}^{\mathbb{T}}$ is generated by $\mathbb{C}$.
It contains $\mathbb{T}$-actions not $\mathrm{KK}^{\mathbb{T}}$-equivalent to an action on a commutative C^*-algebra.

Making objects simple

Theorem

Any object A of KK^G is KK^G-equivalent to a pointwise outer action on a non-zero, simple, purely infinite, stable C^*-algebra B.
If A is nuclear or exact, then so is B.

Proof.

- If $\mathcal{E}$ is a G-equivariant A, A-correspondence, then A is KK^G-equivalent to the Toeplitz algebra $\mathcal{T}_{\mathcal{E}}$. (Pimsner)
- If $\mathcal{E}$ is "infinite", then $\mathcal{T}_{\mathcal{E}}$ is simple and purely infinite. (Kumjian)
- This works if $\mathcal{E} = L^2(G, \mathcal{H}) \otimes A$ with a "regular" covariant representation of A, G on $L^2(G, \mathcal{H})$. □

The Dirac–dual Dirac method for amenable groups

Theorem (Higson–Kasparov)

Let G be an amenable group. There is a nuclear C^*-algebra $\mathcal{P}$ with a proper G-action that is KK^G-equivalent to $\mathbb{C}$.

- This is used by Higson–Kasparov to prove the Baum–Connes conjecture for amenable groups.
- Let $\mathcal{E}G$ be a locally compact, universal proper action of G.

Corollary

Let G be amenable. The following functor is fully faithful:

$$p_{\mathcal{E}G}^*\colon KK^G \to KK^{G\ltimes\mathcal{E}G}, \qquad A \mapsto C_0(\mathcal{E}G, A).$$

one-sided inverse: $KK^{G\ltimes\mathcal{E}G} \to KK^G$, $\mathcal{A} \mapsto \mathcal{P} \otimes_{\mathcal{E}G} \mathcal{A}$

Transferring G-actions to bundles

Proposition

Let G be torsion-free. Let BG be its classifying space.
Then $\mathrm{KK}^{G\ltimes\mathcal{E}G} \cong \mathrm{KK}^{BG}$.

Theorem (Meyer 2019)

Let G be torsion-free and amenable.
Then KK^{G} is equivalent to the subcategory of KK^{BG} consisting of all locally trivial bundles of C^*-algebras over BG.
This equivalence preserves nuclearity.

Bundles of Kirchberg algebras

- Let G be a torsion-free, amenable group.
- Any G-action in the equivariant bootstrap class $\mathfrak{B}^G$ is KK^G-equivalent to one on a stable UCT Kirchberg algebra.
- The equivalence from KK^G to a subcategory of KK^{BG} maps an action on a stable UCT Kirchberg algebra A to a locally trivial bundle with fibre A.
- Kirchberg's classification theorem applies here: KK^{BG}-equivalence $\cong$ isomorphism of bundles.

Theorem (Meyer 2019)

Let G be a torsion-free amenable group, A a stable UCT Kirchberg algebra. There is a bijection
$\{G\text{-actions on } A \text{ in } \mathfrak{B}^G\}/\mathrm{KK}^G\text{-equivalence} \cong$
$\{\mathrm{Aut}(A)\text{-principal bundles over } BG\}/\text{isomorphism}$

Some examples

Example ($G = \mathbb{Z}$)

- The group $\mathbb{Z}$ is torsion-free and amenable.
- $B\mathbb{Z} = \mathbb{T}$.
- $\{\mathrm{Aut}(A)$-principal bundles over $\mathbb{R}/\mathbb{Z}\}$/isomorphism $\cong \pi_0(\mathrm{Aut}(A))$.

Example ($G = \mathbb{Z}^2$)

- The group $\mathbb{Z}^2$ is torsion-free and amenable.
- $B\mathbb{Z}^2 = \mathbb{T}^2$.
- $\{\mathrm{Aut}(A)$-principal bundles over $\mathbb{T}^2\}$/isomorphism $\cong$ homotopy classes of triples (x, y, h), $x, y \in \mathrm{Aut}(A)$, h is a homotopy in $\mathrm{Aut}(A)$ between $x \cdot y$ and $y \cdot x$.

The little invariant

- Let p be a prime and let $G = \mathbb{Z}/p$.
- G-equivariant bootstrap class $\mathfrak{B}^G$ is generated by $\mathbb{C}$, $\mathrm{C}(G)$.

Definition (little invariant for $A \in \mathrm{KK}^G$)

$L_*(A) := \mathrm{KK}^G_*(\mathbb{C}, A) \oplus \mathrm{KK}^G_*(\mathrm{C}(G), A) \cong \mathrm{K}_*(A \rtimes G) \oplus \mathrm{K}_*(A)$m

Proposition

If $A \in \mathfrak{B}^G$, then $A \cong 0$ in KK^G $\iff$ $L_*(A) \cong 0$.

- $L_*(A)$ is a module over $\mathfrak{T} := \mathrm{KK}^G_*(\mathbb{C} \oplus \mathrm{C}(G), \mathbb{C} \oplus \mathrm{C}(G))^{\mathrm{op}}$.

The ring that acts on the little invariant

- The group ring of $\mathbb{Z}/p$ is isomorphic to $\mathbb{Z}[t]/(t^p - 1)$.
- Let $N(t) := 1 + t + t^2 + \cdots + t^{p-1}$.
- So $t^p - 1 = N(t) \cdot (t - 1)$.

Proposition

The ring $\mathfrak{T}$ is concentrated in even parity.
It is isomorphic to the ring of 2×2-matrices

$$\begin{pmatrix} x_{00} & N(t) \cdot x_{01} \\ x_{10} & x_{11} \end{pmatrix}, \quad x_{00}, x_{11} \in \frac{\mathbb{Z}[t]}{(t^p - 1)}, \; x_{01}, x_{10} \in \frac{\mathbb{Z}[t]}{(t - 1)},$$

with the multiplication induced by the multiplication in $\mathbb{M}_2(\mathbb{Z}[t])$.

Köhler's invariant

- Let D be the mapping cone of the inclusion $\mathbb{C} \hookrightarrow \mathrm{C}(G)$.
- $F_*(A) := \mathrm{KK}_*^G(\mathbb{C}, A) \oplus \mathrm{KK}_*^G(\mathrm{C}(G), A) \oplus \mathrm{KK}_*^G(D, A)$.
- Let $\mathfrak{K}$ be the opposite KK_*^G-endomorphism ring of $\mathbb{C} \oplus \mathrm{C}(G) \oplus D$.
- So $F_*(A)$ is always a $\mathbb{Z}/2$-graded module over $\mathfrak{K}$.

Theorem (Köhler's Universal Coefficient Theorem)

- *If* $A \in \mathfrak{B}^G$, $C \in \mathrm{KK}^G$, *there is an extension*

$$\mathrm{Ext}^1_{\mathfrak{K}}\big(F_{1+*}(A), F_*(C)\big) \rightarrowtail \mathrm{KK}_*^G(A, C) \twoheadrightarrow \mathrm{Hom}_{\mathfrak{K}}\big(F_*(A), F_*(C)\big).$$

- *Let* $A_1, A_2 \in \mathfrak{B}^G$.
  $F_*(A_1) \cong F_*(A_2)$ *as graded* $\mathfrak{K}$*-modules* $\Longrightarrow A_1 \cong_{\mathrm{KK}^G} A_2$

The range of Köhler's invariant

- If $A \in \mathrm{KK}^G$, then the Puppe sequence for the inclusion $\mathbb{C} \hookrightarrow \mathrm{C}(G)$ is an exact sequence for the pieces of $F_*(A)$.
- Baaj–Skandalis duality gives an automorphism of KK^G.
- It maps $\mathbb{C} \mapsto \mathrm{C}(G)$, $\mathrm{C}(G) \mapsto \mathbb{C}$, and $D \mapsto \Sigma D$.
- So it induces an automorphism of the ungraded ring $\mathfrak{K}$.
- Baaj–Skandalis duality gives another exact sequence for Köhler's invariant of $A \in \mathrm{KK}^G$.

Theorem

A $\mathfrak{K}$-module is of the form $F_*(A)$ for $A \in \mathfrak{B}^G$ (or $A \in \mathrm{KK}^G$) $\iff$ it is countable and the two sequences above are exact.

The classification problem

- Köhler's results imply that $\{\text{objects of } \mathfrak{B}^G\}/\mathrm{KK}^G\text{-equivalence} \cong \{\text{exact } \mathfrak{K}\text{-modules}\}/\text{isomorphism}$
- Any action on an object of $\mathfrak{B}^G$ is KK^G-equivalent to a pointwise outer action on a UCT stable Kirchberg algebra.
- Köhler already computed the graded ring $\mathfrak{K}$ explicitly.
- I describe it through generators and relations.
- This simplifies describing exact modules over it.

Generators and relations for Köhler's ring

Theorem

The ring $\mathfrak{K}$ is the universal ring generated by elements 1_j for $j = 0, 1, 2$ and α_{jk} for $0 \le j, k \le 2$ with $j \neq k$ with the relations:

$$
\begin{aligned}
1_j \cdot 1_k &= \delta_{j,k} 1_j \qquad \textit{orthogonal idempotents},\\
1_0 + 1_1 + 1_2 &= 1.\\
1_j \cdot \alpha_{jk} \cdot 1_k &= \alpha_{jk},\\
\alpha_{jk} \cdot \alpha_{km} &= 0 \qquad \textit{if } \{j, k, m\} = \{0, 1, 2\},\\
\alpha_{01} \cdot \alpha_{10} &= N(1_0 - \alpha_{02} \cdot \alpha_{20}),\\
\alpha_{10} \cdot \alpha_{01} &= N(1_1 - \alpha_{12} \cdot \alpha_{21}),\\
p \cdot 1_2 &= N(1_2 - \alpha_{20} \cdot \alpha_{02}) + N(1_2 - \alpha_{21} \cdot \alpha_{12}).
\end{aligned}
$$

α_{12} and α_{21} are odd and all other generators are even.

Uniquely p-divisible modules

Definition

Let ϑ be a primitive pth root of unity. $\mathbb{Z}[\vartheta] \cong \mathbb{Z}[t]/(N(t))$.

Let X be a $\mathbb{Z}/2$-graded Abelian group and let Y and Z be two $\mathbb{Z}/2$-graded $\mathbb{Z}[\vartheta]$-modules. Assume that X, Y, Z are uniquely p-divisble. We define an exact $\mathfrak{K}$-module:

$$M_0 := X \oplus Y \qquad M_1 := X \oplus Z \qquad M_2 := Y \oplus \Sigma Z$$

$$\alpha_{01}^M = \begin{pmatrix} 1^X & 0 \\ 0 & 0 \end{pmatrix} \quad \alpha_{12}^M = \begin{pmatrix} 0 & 0 \\ 0 & (1-\vartheta)^Z \end{pmatrix} \quad \alpha_{20}^M = \begin{pmatrix} 0 & 1^Y \\ 0 & 0 \end{pmatrix}$$

$$\alpha_{10}^M = \begin{pmatrix} p^X & 0 \\ 0 & 0 \end{pmatrix} \quad \alpha_{21}^M = \begin{pmatrix} 0 & 0 \\ 0 & 1^Z \end{pmatrix} \quad \alpha_{02}^M = \begin{pmatrix} 0 & 0 \\ (1-\vartheta)^Y & 0 \end{pmatrix}$$

Here ΣZ means Z with opposite parity.

Structure of uniquely p-divisible exact modules

Theorem

Let M be an exact $\mathfrak{K}$-module where M_0, M_1 or M_2 is uniquely p-divisible.
Then the other two pieces are uniquely p-divisible.
There are modules X over $\mathbb{Z}[1/p]$ and Y, Z over $\mathbb{Z}[\vartheta, 1/p]$ such that M is isomorphic to the exact $\mathfrak{K}$-module built above.

- A G-action on $\mathcal{O}_n$ gives an exact $\mathfrak{K}$-module with $M_1 \cong \mathrm{K}_*(\mathcal{O}_n) \cong \mathbb{Z}/(n+1)$.
- $\mathbb{Z}/(n+1)$ uniquely p-divisible $\iff (p, n+1) = 1$
- We assume $(p, n+1) = 1$.
- The pieces X, Z above are determined by the induced G-action on $\mathrm{K}_*(\mathcal{O}_n)$.
- Given this, the actions in $\mathfrak{B}^G$ up to KK^G-equivalence are in bijection with $\mathbb{Z}[\vartheta, 1/p]$-modules.

The general case

- Exact modules over $\mathfrak{K}$ with cyclic M_1 reduce to $M_1 = \mathbb{Z}$ or $M_1 = \mathbb{Z}/p^k$.
- I find a list of 10 (series of) examples of such modules, such that any other exact $\mathfrak{K}$-module with cyclic M_1 is an extension of one of these examples by a uniquely p-divisible one.

Example (the trivial action on $\mathcal{O}_\infty$)

$M_0 := \mathfrak{S} = \mathbb{Z}[t]/(t^p - 1)$,
$M_1 := \mathbb{Z}$,
$M_2 := \mathfrak{S}/(N(t)) \cong \mathbb{Z}[\vartheta]$,
all in even degree. Let $\alpha_{21}^M := 0$ and $\alpha_{12}^M := 0$. Let $\alpha_{01}^M \colon M_1 \to M_0$ be multiplication with $N(t)$. Let α_{10}^M be evaluation at 1.
Let α_{20}^M be the quotient map.
Let α_{02}^M be induced by multiplication with $(1 - t)$.

Summary

- If G is an amenable group without torsion: {group actions on a stable UCT Kirchberg algebra A}/KK^G-equivalence $\cong$ {$\mathrm{Aut}(A)$-principal bundles over BG}/isomorphism
- If the order of G is prime, then there is a Universal Coefficient Theorem for G-actions in the bootstrap class.
- We can classify G-actions in the bootstrap class up to KK^G-equivalence.
- Many actions are uniquely p-divisible.
- Up to that, there is a finite, but complicated list of series of examples.