INTRODUCTION TO QUANTUM GROUPS

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Abstract. These notes contain an introduction to the theory of quantum groups in the setting of operator algebras.

We start with some background on finite dimensional Hopf algebras, and introduce key concepts in the theory of locally compact quantum groups, such as the Kac-Takesaki operator, in this basic setting. Then we present the definition of a locally compact quantum group in the sense of Kustermans-Vaes, and list some fundamental results regarding the structure of locally compact quantum groups, without proof.

In the second part of the notes we focus our attention on the special case of compact quantum groups. Following the pioneering work of Woronowicz, we give a self-contained account to this class of quantum groups. In particular, we discuss the representation theory of compact quantum groups, leading up to the Peter-Weyl theorem.

The last part of the notes is devoted to connections between quantum groups and noncommutative geometry. We have picked one specific example to illustrate these links, namely the spectral geometry of the standard Podleś sphere. We describe the spectral triple introduced to Dąbrowski-Sitarz, deforming the Riemann sphere and its standard Dirac operator, and discuss some of its applications.

1. Introduction

Quantum groups are a mathematical structure at the interface of mathematical physics, representation theory, operator algebras, and noncommutative geometry. Historically, they arose from the inverse scattering method, and from efforts to generalise Pontrjagin duality beyond the case of abelian locally compact groups. Some landmark contributions are the ICM paper by Drinfeld [9], the work on $SU_q(2)$ by Woronowicz [32], and the theory of locally compact quantum groups due to Kustermans-Vaes [13]. By now, there is a vast literature on the subject, including several excellent textbooks [17], [4], [19], [11], [23], with focus on different aspects of the theory.

These notes aim to give a basic introduction to quantum groups, with emphasis on the operator algebraic approach to the theory, and connections with noncommutative geometry. In the first part we discuss the duality of finite dimensional Hopf $*$-algebras and the closely related notion of a finite quantum group. At the outset, the advantage of working in this restricted setting is that some notorious technical difficulties disappear, and the key ideas are easy to motivate. After this warmup, we move directly to the definition of a locally compact quantum group, using the notion of a Hopf C^*-algebra as a key ingredient. However, we do not develop any of the theory of general locally compact quantum groups beyond the definition. If one wanted to pursue this direction, it would in fact be more natural to work in the von Neumann algebraic setting [14]. For our purposes however, the C^*-algebraic setting is a convenient choice since we shall focus mainly on compact quantum groups in the remaining part of the notes. The definition of a compact quantum group, due to Woronowicz, is both elegant and concise, and leads to a rich and interesting theory. This is the subject of the second part of the notes. We cover the theory of compact quantum groups in detail, following the standard sources [23], [33], [18], [11]. Our presentation is slightly different, which allows one to move to the main results very directly. In the third part we discuss connections between quantum groups and noncommutative geometry. This interplay has a long history, and it is still an active area of research. One of the fundamental challenges is that the notion of a spectral triple [6], [7] does not interact particularly well with the type of deformations which are ubiquitous in quantum group theory. These difficulties have been overcome in a few cases however, and we will consider a particularly nice example to illustrate this, namely the Podleś sphere [8].

We have also included a few remarks on the general concept of quantum symmetry. The idea is that the correct notion of symmetry in noncommutative geometry should incorporate effects which go beyond ordinary automorphisms. Remarkably, this idea has led to interesting connections with other areas, including the study of subfactors, free probability, and quantum information.

Throughout the text we assume familiarity with basic concepts and results from the theory of C^*-algebras, but we do not expect any prior knowledge about Hopf algebras or quantum groups. For the last section, the reader should be accustomed with ideas from noncommutative geometry, most importantly the notion of a spectral triple. We have however aimed to give precise references to more advanced material which is used without proof.

These notes are organised as follows. In Section 2 we discuss the theory of finite dimensional Hopf $*$-algebras and finite quantum groups, thus laying some groundwork motivating the definition of a locally compact quantum group. As already pointed out above, working in the finite setting has the advantage that the basic ideas become particularly transparent. Section 3 contains a quick introduction to the general theory of locally compact quantum groups, starting from the concept of a Hopf $*$-algebra. We sketch some of the main structure results about these objects, which can be viewed as direct analogues of basic facts on finite quantum groups, albeit requiring much more sophisticated proofs. For the details we refer directly to the original work of Kustermans-Vaes. Then, in Section 4, we focus our attention on the case of compact quantum groups. Many of the difficulties in the general locally compact setting simplify quite significantly in this case, and we develop the theory of these quantum groups from scratch. We also introduce a number of prominent examples of compact quantum groups, including the quantum group $SU_q(2)$ of Woronowicz, free unitary quantum groups, free orthogonal quantum groups, and quantum permutation groups. In Section 5 we present some general definitions and facts from the representation theory of locally compact quantum groups. Building on this, we discuss the representation theory of compact quantum groups in Section 6. Section 7 explains now to define actions of quantum groups on C^*-algebras. In the same way as for classical groups, the study of such actions is key to the understanding of locally compact quantum groups, and many examples are directly motivated by an attempt to find generalised symmetries of certain C^*-algebras. Finally, Section 8 is devoted to the noncommutative geometry of $SU_q(2)$ and the standard Podleś sphere $SU_q(2)/T$. We present the spectral triple of Dąbrowski-Sitarz and discuss some of its properties, and sketch an application to the Baum-Connes conjecture.

Let us conclude with some remarks on notation. We write $B(\mathcal{E})$ for the algebra of bounded operators on a Hilbert space, or more generally, the adjointable operators on a Hilbert A-module $\mathcal{E}$ for some C^*-algebra A. The algebra of compact operators on $\mathcal{E}$ is denoted by $K(\mathcal{E})$. We write $[X]$ for the closed linear span of a subset X of a Banach space. Depending on the context, the symbol $\otimes$ denotes either the tensor product of Hilbert spaces or the minimal tensor product of C^*-algebras. We will sometimes write $\odot$ for algebraic tensor products, mainly when it seems helpful to distinguish them from completed tensor products. The ground field in the algebraic part is always chosen to be $\mathbb{C}$, although many definitions and results work more generally.

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2. Finite quantum groups

Following the general philosophy of noncommutative geometry [5], a "noncommutative space", or "quantum space", is encoded by its noncommutative algebra of "functions". In particular, in the spirit of Gelfand duality, a noncommutative locally compact space is the same thing as a C^*-algebra.

Now a classical locally compact group is a locally compact space equipped with additional structure, reflected in the group multiplication, the unit element, and the fact that every group element has an inverse. Moreover, the group operations are required to be compatible with the topology. We should therefore expect a "locally compact quantum group" to be given by a C^*-algebra, encoding the underlying noncommutative space, together with extra data which reflects the "group" structure. Indeed, this is essentially how the theory of locally compact quantum groups works; however the extra structure in the quantum case is more intricate than in the classical situation. A key feature is that if the underlying noncommutative space is classical, that is, the function algebra of a locally compact space, then everything reduces precisely to the familiar concepts from group theory.

Before entering the discussion of general locally compact quantum groups, we develop the theory of finite quantum groups in this section, making the above ideas precise in the special case of finite groups. This captures already many of the structures which appear in the general case.

In order to get started let G be a finite group. Then the group structure of G can be encoded at the level of the algebra $C(G)$ of complex-valued functions on G. More precisely, we have

- the comultiplication $\Delta : C(G) \to C(G) \otimes C(G) = C(G \times G)$ given by
$$\Delta(f)(s,t) = f(st),$$
- the antipode $S : C(G) \to C(G)$ given by
$$S(f)(t) = f(t^{-1}),$$
- the counit $\epsilon : C(G) \to \mathbb{C}$ given by
$$\epsilon(f) = f(e),$$
where $e \in G$ is the identity element.

These maps satisfy a number of conditions, which can be summarised by saying that they turn $C(G)$ into a *Hopf algebra*.

Definition 2.1. *A Hopf algebra is a unital algebra H together with*

a) a unital algebra homomorphism $\Delta : H \to H \otimes H$, called comultiplication,
b) a unital algebra homomorphism $\epsilon : H \to \mathbb{C}$, called counit,
c) a linear map $S : H \to H$, called antipode,

such that

a) (Coassociativity) $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$,
b) (Counitality) $(\epsilon \otimes \mathrm{id})\Delta = \mathrm{id} = (\mathrm{id} \otimes \epsilon)\Delta$,
c) (Antipode axiom) We have
$$\mu(S \otimes \mathrm{id})\Delta(f) = \epsilon(f)1 = \mu(\mathrm{id} \otimes S)\Delta(f)$$
for all $f \in H$, where $\mu : H \otimes H \to H$ is the multiplication map.

Here we always work over the complex numbers, but the definitions make sense over any field, or even any commutative ring. The theory of Hopf algebras has a long history, and it originated from considerations in algebraic topology. Standard references in the purely algebraic literature include [1], [21].

The antipode in a Hopf algebra is typically not an algebra homomorphism, and it is rather a unital algebra antihomomorphism. That is, we always have $S(fg) = S(g)S(f)$ for $f, g \in H$ and $S(1) = 1$. If $\sigma : H \otimes H \to H \otimes H, \sigma(f \otimes g) = g \otimes f$ denotes the flip map, then the antimultiplicativity of S can be phrased as $\mu \circ (S \otimes S) \circ \sigma = S \circ \mu$. Similarly, the comultiplication Δ satisfies

$$\sigma \circ \Delta \circ S = (S \otimes S) \circ \Delta$$

and $\epsilon \circ S = \epsilon$. This can be interpreted as saying that S is a coalgebra antihomomorphism. For proofs of these facts see for instance [11, Section 1.2.4].

We will be interested in Hopf algebras which carry a compatible $*$-structure in the following sense.

Definition 2.2. *A Hopf $*$-algebra is a Hopf algebra H whose underlying algebra is $*$-algebra such that the coproduct $\Delta : H \to H \otimes H$ is a $*$-homomorphism.*

In a Hopf $*$-algebra H the counit ϵ is always a $*$-homomorphism, and the antipode satisfies the relation

$$S(S(f)^*)^* = f$$

for all $f \in H$, compare [11, Section 1.2.7]. This means in particular that the antipode S in a Hopf $*$-algebra is invertible, with inverse S^{-1} given by $S^{-1}(f) = S(f^*)^*$. We note that the antipode in a general Hopf algebra need not be invertible, although this property holds in most examples of interest.

Exercise 1. *Let H be a Hopf $*$-algebra. Show that H^{cop}, which is H with the same algebra structure, unit element and counit, but the flipped comultiplication*

$$\Delta^{\mathsf{cop}} = \sigma \circ \Delta,$$

is a Hopf $*$-algebra with antipode S^{-1}.

Let us consider some basic examples of Hopf $*$-algebras.

Example 2.3. *Let G be a finite group.*
a) The algebra $C(G)$ of functions on G with the maps described further above is a Hopf $*$-algebra.
b) The group algebra $\mathbb{C}[G] = C^*(G)$ of G is a Hopf $*$-algebra with the structure maps determined by

$$\widehat{\Delta}(u_t) = u_t \otimes u_t, \qquad \widehat{\epsilon}(u_t) = 1, \qquad \widehat{S}(u_t) = u_{t^{-1}}$$

on the basis elements $u_t \in C^*(G)$ associated with $t \in G$.

In the sequel we will use the Sweedler notation

$$\Delta(f) = f_{(1)} \otimes f_{(2)}$$

for the coproduct of an element f in a Hopf algebra. For instance, the antipode axiom then reads

$$f_{(1)}S(f_{(2)}) = \epsilon(f) = S(f_{(1)})f_{(2)}.$$

We may also write iterated coproducts in this way, that is,

$$(\Delta \otimes \mathrm{id})\Delta(f) = f_{(1)} \otimes f_{(2)} \otimes f_{(3)} = (\mathrm{id} \otimes \Delta)\Delta(f),$$

noting that the comultiplications can be applied in any order due to the coassociativity axiom.

Let H be a finite dimensional Hopf $*$-algebra. Then the linear dual space $\widehat{H} = \mathrm{Hom}(H, \mathbb{C})$ becomes again a finite dimensional Hopf $*$-algebra by dualising the structure maps of H. More precisely, if $(x, f) = x(f)$ is the canonical pairing of $x \in \widehat{H}$ and $f \in H$ then the product and coproduct of $\widehat{H}$ are defined such that

$$(xy, f) = (x, f_{(1)})(y, f_{(2)}), \qquad (x, fg) = (x_{(2)}, f)(x_{(1)}, g),$$

moreover the antipode $\widehat{S}$, counit $\widehat{\epsilon} : \widehat{H} \to \mathbb{C}$ and unit element $1 \in \widehat{H}$ are determined by

$$(\widehat{S}(x), f) = (x, S^{-1}(f)), \qquad (\widehat{S}^{-1}(x), f) = (x, S(f))$$

and

$$\widehat{\epsilon}(x) = (x, 1), \qquad \epsilon(f) = (1, f)$$

for $x, y \in \widehat{H}$ and $f, g \in H$. Here we use the Sweedler notation $\widehat{\Delta}(x) = x_{(1)} \otimes x_{(2)}$ for $x \in \widehat{H}$. Moreover we obtain a $*$-structure on $\widehat{H}$ by setting

$$(x^*, f) = \overline{(x, S(f)^*)}$$

for $x \in \widehat{H}, f \in H$.

Let us point out straight away that our choice of the coproduct on the dual differs from the usual conventions in the literature on Hopf algebras. More precisely, we swap the order of the tensor factors in the coproduct, whereas it is standard to work with $\widehat{\Delta}^{\mathsf{cop}} = \sigma \circ \widehat{\Delta}$, in our notation. However, our choice is more natural from the point of view of locally compact quantum groups. In fact, even in the case of classical groups it turns out to match better with Fourier duality.

Exercise 2. *Let H be a finite dimensional Hopf $*$-algebra. Show that $\widehat{H}$ with the above structure maps becomes a Hopf $*$-algebra.*

A key feature of the passage from H to the dual $\widehat{H}$ is that it remembers the original Hopf $*$-algebra. This is a consequence of the following fundamental fact.

Theorem 2.4 (Biduality)**.** *Let H be a finite dimensional Hopf $*$-algebra. Then the dual of $\widehat{H}$ is isomorphic to H as a Hopf $*$-algebra.*

Proof. Consider the linear map $I : H \to \widehat{\widehat{H}} = H^{**}$ defined by

$$(I(f), x) = I(f)(x) = (x, S(f))$$

for $x \in \widehat{H}$. Since S is bijective this map is a linear isomorphism.

Now let $f, g \in H$ and $x, y \in \widehat{H}$ be arbitrary. By definition of the pairing between $\widehat{\widehat{H}}$ and $\widehat{H}$ we obtain

$$(I(fg), x) = (x, S(fg)) = (x, S(g)S(f)) = (x_{(1)}, S(f))(x_{(2)}, S(g)) = (I(f)I(g), x),$$

so that $I(fg) = I(f)I(g)$. Similarly,

$$\begin{aligned}(I(f^*), x) = (x, S(f^*)) &= \overline{(x^*, f)} \\ &= \overline{(\hat{S}^{-1}(x^*), S(f))} \\ &= \overline{(\hat{S}(x)^*, S(f))} = \overline{(I(f), \widehat{S}(x)^*)} = (I(f)^*, x)\end{aligned}$$

so that $I(f^*) = I(f)^*$, and

$$\begin{aligned}(I(f)_{(1)} \otimes I(f)_{(2)}, x \otimes y) = (I(f), yx) &= (yx, S(f)) \\ &= (y, S(f_{(2)})(x, S(f_{(1)})) \\ &= (I(f_{(1)}) \otimes I(f_{(2)}), x \otimes y),\end{aligned}$$

which means $(I \otimes I)\Delta = \widehat{\widehat{\Delta}} I$. Finally, we have $(I(1), x) = (x, 1) = \widehat{\epsilon}(x) = (1, x)$ and

$$\widehat{\widehat{\epsilon}}(I(f)) = (I(f), 1) = (1, S(f)) = \epsilon(f).$$

Hence I is compatible with the Hopf $*$-algebra structures on H and $\widehat{\widehat{H}}$. ☐

Let us next discuss *Haar functionals*. A *left Haar functional* for a Hopf algebra H is a linear functional $\phi : H \to \mathbb{C}$ such that

$$(\mathrm{id} \otimes \phi)\Delta(f) = \phi(f)1$$

for all $f \in H$. Similarly, a *right Haar functional* for H is a linear functional $\psi : H \to \mathbb{C}$ such that

$$(\psi \otimes \mathrm{id})\Delta(f) = \psi(f)1$$

for all $f \in H$.

Exercise 3. *Let H be a Hopf algebra with invertible antipode. Show that if ϕ is a left Haar functional for H then $\psi(f) = \phi(S(f))$ is a right Haar functional.*

Let H be a Hopf algebra with invertible antipode, and assume that ϕ is a left Haar functional for H with $\phi(1) \neq 0$. We claim that ϕ is also a right Haar functional. To shows this we may assume without loss of generality that $\phi(1) = 1$. If we consider the right Haar functional $\psi = \phi \circ S$, compare Exercise 3, we get

$$\phi(f) = \phi(f)\psi(1) = (\psi \otimes \phi)\Delta(f) = \psi(f)\phi(1) = \psi(f)$$

for all $f \in H$, which means in particular that ϕ is a right Haar functional.

Recall that a linear functional $\omega : A \to \mathbb{C}$ on a $*$-algebra A is called *positive* if $\omega(f^* f) \geq 0$ for all $f \in A$. Moreover, if A is unital then a positive linear functional ω on A is called a *state* if $\omega(1) = 1$. If a Hopf $*$-algebra admits a (left and right) invariant Haar functional which is a state, we shall refer to this as a *Haar state*. Note here that the antipode in a Hopf $*$-algebra is bijective, so that the above discussion applies.

Exercise 4. *Show that a Haar state on a Hopf $*$-algebra is unique if it exists.*

Let us revisit our examples of Hopf $*$-algebras.

Example 2.5. *a) If $H = C(G)$ for a finite group G, the functional*

$$\phi(f) = \frac{1}{|G|} \sum_{t \in G} f(t)$$

is a Haar state.
b) If $H = C^*(G)$ for a finite group G, the functional

$$\widehat{\phi}\left(\sum_{t \in G} c_t u_t \right) = c_e$$

is a Haar state.

The Hopf $*$-algebras in Example 2.3 are in fact finite quantum groups in the following sense.

Definition 2.6. *A finite quantum group is a Hopf $*$-algebra which admits a faithful Haar state.*

Here we recall that a state ω on a $*$-algebra A is called faithful if $\omega(f^* f) = 0$ for $f \in A$ implies $f = 0$. It turns out that a Haar state on a finite dimensional Hopf $*$-algebra is automatically faithful. This follows from the fact that every finite dimensional Hopf algebra is a Frobenius algebra, see [21, Theorem 2.1.3].

If H is a Hopf $*$-algebra with a positive faithful Haar state we will formally write $H = C(G)$ in analogy to the case of finite groups, and by slight abuse of terminology say that G “is” a finite quantum group, rather than H. We will think of H as the algebra of functions on the “virtual” object G, and view the dual Hopf $*$-algebra $\widehat{H} = C^*(G)$ as the group algebra of G.

Since $\widehat{H} = C^*(G)$ is the linear dual of $H = C(G)$ and ϕ is faithful, we can define a linear isomorphism $\mathcal{F} : C(G) \to C^*(G)$ by setting

$$\mathcal{F}(f)(h) = \phi(hf)$$

for $f, g \in H$. As we will see in Lemma 2.8 below, this map can be viewed as an analogue of the Fourier transform.

Exercise 5. *Consider $H = C(G)$ for a finite quantum group G. Show that $\widehat{H} = C^*(G)$ admits a left Haar functional $\widehat{\phi}$ given by*

$$\widehat{\phi}(\mathcal{F}(f)) = \epsilon(f)$$

for all $f \in H$.

We will show next that the functional $\widehat{\phi}$ is positive and faithful, so that $\widehat{H}$ becomes a finite quantum group.

Proposition 2.7. *Keeping the notation from above, we have*

$$\widehat{\phi}(\mathcal{F}(f)^*\mathcal{F}(g)) = \phi(f^*g)$$

for all $f, g \in H$. In particular, the functional $\widehat{\phi}$ on $\widehat{H}$ is positive and faithful, and the Hopf $*$-algebra $\widehat{H}$ is a finite quantum group.

Proof. For $f, g, h \in C(G)$ we compute

$$\begin{aligned}
(\mathcal{F}(f)^*\mathcal{F}(g), h) &= (\mathcal{F}(f)^*, h_{(1)})(\mathcal{F}(g), h_{(2)}) \\
&= (\mathcal{F}(f)^*, h_{(1)})\phi(h_{(2)}g) \\
&= (\mathcal{F}(f)^*, h_{(1)}g_{(2)}S^{-1}(g_{(1)}))\phi(h_{(2)}g_{(3)}) \\
&= (\mathcal{F}(f)^*, S^{-1}(g_{(1)}))\phi(hg_{(2)}) \\
&= \overline{\phi(g_{(1)}^* f)}\phi(hg_{(2)}) \\
&= \phi(f^*g_{(1)})(\mathcal{F}(g_{(2)}), h),
\end{aligned}$$

using invariance of ϕ and the fact that ϕ is positive. Hence we obtain

$$\widehat{\phi}(\mathcal{F}(f)^*\mathcal{F}(g)) = \phi(f^*g_{(1)})\widehat{\phi}(\mathcal{F}(g_{(2)})) = \phi(f^*g_{(1)})\epsilon(g_{(2)}) = \phi(f^*g),$$

as required.

This calculation shows that $\widehat{\phi}(\mathcal{F}(f)^*\mathcal{F}(f)) = \phi(f^*f) \geq 0$ for all $f \in H = C(G)$, and $\widehat{\phi}(\mathcal{F}(f)^*\mathcal{F}(f)) = 0$ iff $f = 0$. Since every element of $\widehat{H}$ is of the form $\mathcal{F}(f)$ for some $f \in H$ this means that $\widehat{\phi}$ is a faithful positive linear functional on $\widehat{H}$.

Now using that $\widehat{\phi}$ is faithful we get $\widehat{\phi}(1) = \widehat{\phi}(1^*1) > 0$, and it follows from our discussion further above that $\widehat{\phi}$ is also right invariant. Upon dividing by $\widehat{\phi}(1)$, we can normalise $\widehat{\phi}$ to obtain a faithful Haar state on $\widehat{H}$. This means that $\widehat{H}$ is a finite quantum group. □

If G is a finite *abelian* group one can show that $\widehat{H}$ identifies, as a finite quantum group, with the algebra of functions $C(\widehat{G})$ where $\widehat{G}$ is the *Pontrjagin dual group* of G, compare [10, Chapter 4]. In the sequel we shall write $\widehat{H} = C(\widehat{G})$ also for a general finite quantum group $H = C(\widehat{G})$, and call $\widehat{G}$ the *dual quantum group* of G.

The positive Haar functional $\widehat{\phi}$ on $C(\widehat{G})$ defined above satisfies $\widehat{\phi}(\phi) = \widehat{\phi}(\mathcal{F}(1)) = 1$. Let us point out that $\widehat{\phi}$ fails to be a state. More precisely, it can be shown that $\widehat{\phi}(1) = \dim(H)$, see for instance [27, Proposition 3.4 and Theorem 3.14].

In order to justify why our normalisation of $\widehat{\phi}$ is natural let us consider the GNS-construction for the Haar state $\phi : C(G) \to \mathbb{C}$. We will denoted the corresponding Hilbert space by $L^2(G)$. Since $C(G)$ is finite dimensional and ϕ is faithful, the construction of $L^2(G)$ amounts to viewing $C(G)$ as a Hilbert space with the scalar product

$$\langle \Lambda(f), \Lambda(g) \rangle = \phi(f^* g)$$

for $f, g \in C(G)$. Here we explicitly write the GNS-map $\Lambda : C(G) \to L^2(G)$ in order to distinguish elements of the algebra $C(G)$ from elements of the Hilbert space $L^2(G)$, but otherwise Λ is nothing but the identity map in disguise.

Similarly, we obtain the GNS-construction for $\widehat{\phi} : C(\widehat{G}) \to \mathbb{C}$, leading to the Hilbert space $L^2(\widehat{G})$. We write $\widehat{\Lambda} : C(\widehat{G}) \to L^2(\widehat{G})$ for the corresponding GNS-map.

Lemma 2.8. *Let G be a finite quantum group. The Fourier transform $\mathcal{F} : C(G) \to C^*(G)$, given by $\mathcal{F}(f)(h) = \phi(hf)$, induces a unitary isomorphism $L^2(G) \to L^2(\widehat{G})$.*

Proof. This follows immediately from Lemma 2.7. Indeed, we have

$$\langle \widehat{\Lambda}(\mathcal{F}(f)), \widehat{\Lambda}(\mathcal{F}(g)) \rangle = \widehat{\phi}(\mathcal{F}(f)^* \mathcal{F}(g)) = \phi(f^* g) = \langle f, g \rangle$$

for all $f, g \in H$, which shows that $\mathcal{F}$ defines an isometric isomorphism with respect to the scalar products on $L^2(G)$ and $L^2(\widehat{G})$, respectively. $\square$

Let us now study the structure of G from the Hilbert space perspective. We obtain a linear operator $W^* \in B(L^2(G) \otimes L^2(G))$ by setting

$$W^*(\Lambda(f) \otimes \Lambda(g)) = (\Lambda \otimes \Lambda)(\Delta(g)(f \otimes 1)) = \Lambda(g_{(1)} f) \otimes \Lambda(g_{(2)})$$

for all $f, g \in C(G)$. As we shall see, this operator encodes essentially all the information of our quantum group and its dual.

Lemma 2.9. *The operator W^* is unitary, with inverse given by*

$$W(\Lambda(f) \otimes \Lambda(g)) = \Lambda(S^{-1}(g_{(1)}) f) \otimes \Lambda(g_{(2)})$$

for $f, g \in H$.

Proof. Using the invariance property of ϕ we calculate

$$\begin{aligned}
\langle W^*(\Lambda(f) \otimes \Lambda(g)), W^*(\Lambda(h) \otimes \Lambda(k)) \rangle &= \langle \Lambda(g_{(1)} f) \otimes \Lambda(g_{(2)}), \Lambda(k_{(1)} h) \otimes \Lambda(k_{(2)}) \rangle \\
&= \phi(f^* g^*_{(1)} k_{(1)} h) \phi(g^*_{(2)} k_{(2)}) \\
&= \phi(f^* h) \phi(g^* k) \\
&= \langle \Lambda(f) \otimes \Lambda(g), \Lambda(h) \otimes \Lambda(k) \rangle,
\end{aligned}$$

which shows that W^* is an isometry. Since $L^2(G)$ is finite dimensional it follows that W^* is in fact a unitary. To verify the formula for its adjoint W, we calculate

$$\begin{aligned}
W^* W(\Lambda(f) \otimes \Lambda(g)) &= W^*(\Lambda(S^{-1}(g_{(1)}) f) \otimes \Lambda(g_{(2)})) \\
&= \Lambda(g_{(2)} S^{-1}(g_{(1)}) f) \otimes \Lambda(g_{(3)}) \\
&= \epsilon(g_{(1)}) \Lambda(f) \otimes \Lambda(g_{(2)}) = \Lambda(f) \otimes \Lambda(g)
\end{aligned}$$

for all f, g, using the antipode and counit axioms for $C(G)^{\mathsf{cop}}$. $\square$

The operator $W \in B(L^2(G) \otimes L^2(G))$ from Lemma 2.9 is called the *Kac-Takesaki operator* of G.

In the sequel we will use the *leg numbering notation* for operators on multiple tensor products. For instance, we write W_{12} for the operator $W \otimes \mathrm{id} \in B(L^2(G) \otimes L^2(G) \otimes L^2(G))$, and similarly $W_{23} = \mathrm{id} \otimes W$. In addition,

$$W_{13} = \Sigma_{12} W_{23} \Sigma_{12} = \Sigma_{23} W_{12} \Sigma_{23}$$

where $\Sigma \in B(L^2(G) \otimes L^2(G))$ is the tensor flip. In other words, the indices specify in which tensor factors an operator acts.

Exercise 6. *Check that the Kac-Takesaki operator W is a multiplicative unitary, that is, show that the pentagon equation*

$$W_{12}W_{13}W_{23} = W_{23}W_{12}$$

holds in $B(L^2(G) \otimes L^2(G) \otimes L^2(G))$.

The action of $C(G)$ on itself by left multiplication induces a unital $*$-homomorphism $\widehat{\lambda} : C(G) \to B(L^2(G))$, given by

$$\widehat{\lambda}(f)(\Lambda(g)) = \Lambda(fg)$$

for $f, g \in C(G)$. Since ϕ is assumed to be faithful the map λ is injective, and we will often identify $C(G)$ with a $*$-subalgebra of $B(L^2(G))$. Note in particular that $C(G)$ is a C^*-algebra.

Lemma 2.10. *We have*

$$\Delta(f) = W^*(1 \otimes f)W$$

for all $f \in C(G)$.

Proof. We compute

$$\begin{aligned}
(W^*(1 \otimes f)W)(\Lambda(g) \otimes \Lambda(h)) &= (W^*(1 \otimes f))(\Lambda(S^{-1}(h_{(1)})g) \otimes \Lambda(h_{(2)})) \\
&= W^*(\Lambda(S^{-1}(h_{(1)})g)) \otimes \Lambda(fh_{(2)})) \\
&= \Lambda(f_{(1)}g) \otimes \Lambda(f_{(2)}h) \\
&= \Delta(f)(\Lambda(g) \otimes \Lambda(h))
\end{aligned}$$

for all $g, h \in C(G)$. This yields the claim. □

Our next goal is to describe the dual $C^*(G) = C(\widehat{G})$ inside $B(L^2(G))$. Using the Fourier transform from Lemma 2.8 we can transport the left regular representation of $C(\widehat{G}) = C^*(G)$ on $L^2(\widehat{G})$ to $L^2(G)$ as follows. Let us define a linear map λ from $C^*(G)$ to $B(L^2(G))$ by the formula

$$\lambda(x)\Lambda(f) = (\hat{S}(x), f_{(1)})\Lambda(f_{(2)}) = (x, S^{-1}(f_{(1)}))\Lambda(f_{(2)}),$$

for $x \in C^*(G), f \in C(G)$. Then for all $h \in C(G)$ we have

$$\begin{aligned}
(\mathcal{F}(\lambda(x)\Lambda(f)), h) &= (x, S^{-1}(f_{(1)}))\phi(hf_{(2)}) \\
&= (x, S^{-1}(S(h_{(1)})h_{(2)}f_{(1)}))\phi(h_{(3)}f_{(2)}) \\
&= (x, h_{(1)})\phi(h_{(2)}f) \\
&= (x, h_{(1)})(\mathcal{F}(f), h_{(2)}) \\
&= (x\mathcal{F}(f), h),
\end{aligned}$$

which means that $\mathcal{F}\lambda(x)\mathcal{F}^{-1}$ corresponds to the GNS-representation of $C(\widehat{G})$ on $L^2(\widehat{G})$. In particular, the above formula yields a faithful $*$-representation $\lambda : C^*(G) \to B(L^2(G))$, and we we may identify $C^*(G)$ with a $*$-subalgebra of $B(L^2(G))$ in this way.

Exercise 7. *The comultiplication $\widehat{\Delta}$ for $C^*(G)$ is given by*

$$\widehat{\Delta}(x) = \widehat{W}^*(1 \otimes x)\widehat{W}$$

where $\widehat{W} = \Sigma W^* \Sigma$. Here Σ denotes the flip map on $L^2(G) \otimes L^2(G)$.

Let us verify that $\widehat{W} = \Sigma W^* \Sigma$ corresponds indeed to the Kac-Takesaki operator of $\widehat{G}$ under Fourier transform. To this end note that for any elements $f, g \in C(G)$ we obtain

$$\begin{aligned}(\mathcal{F} \otimes \mathcal{F})(\Sigma W^* \Sigma)(\Lambda(f) \otimes \Lambda(g)) &= (\mathcal{F} \otimes \mathcal{F})(\Lambda(f_{(2)}) \otimes \Lambda(f_{(1)} g)) \\ &= \widehat{\Lambda}(\mathcal{F}(f_{(2)})) \otimes \widehat{\Lambda}(\mathcal{F}(f_{(1)} g)).\end{aligned}$$

Now, for $h, k \in C(G)$ we compute

$$\begin{aligned}(\mathcal{F}(f_{(2)}) \otimes \mathcal{F}(f_{(1)} g), h \otimes k) &= \phi(h f_{(2)}) \phi(k f_{(1)} g) \\ &= \phi(h_{(3)} f_{(2)}) \phi(k S(h_{(1)}) h_{(2)} f_{(1)} g) \\ &= \phi(k S(h_{(1)}) g) \phi(h_{(2)} f) \\ &= (\mathcal{F}(g), k S(h_{(1)}))(\mathcal{F}(f), h_{(2)}) \\ &= (\mathcal{F}(g)_{(1)}, S(h_{(1)}))(\mathcal{F}(f), h_{(2)})(\mathcal{F}(g)_{(2)}, k) \\ &= (\hat{S}^{-1}(\mathcal{F}(g)_{(1)}) \mathcal{F}(f) \otimes \mathcal{F}(g)_{(2)}, h \otimes k),\end{aligned}$$

and therefore the previous calculation gives indeed

$$\begin{aligned}(\mathcal{F} \otimes \mathcal{F})(\Sigma W^* \Sigma)(\Lambda(f) \otimes \Lambda(g)) &= \widehat{\Lambda}(\mathcal{F}(f_{(2)})) \otimes \widehat{\Lambda}(\mathcal{F}(f_{(1)} g)) \\ &= (\widehat{\Lambda} \otimes \widehat{\Lambda})(\widehat{S}^{-1}(\mathcal{F}(g)_{(1)}) \mathcal{F}(f) \otimes \mathcal{F}(g)_{(2)}) \\ &= \widehat{W}(\widehat{\Lambda}(\mathcal{F}(f)) \otimes \widehat{\Lambda}(\mathcal{F}(g))).\end{aligned}$$

One can check that the algebra $H = C(G) \subset B(L^2(G))$ consists precisely of all operators $(\mathrm{id} \otimes \omega)(W)$ for $\omega \in B(L^2(G))^*$. Similarly, one finds that $\widehat{H} = C^*(G) \subset B(L^2(G))$ identifies with the set of all operators of the form $(\omega \otimes \mathrm{id})(W)$ for $\omega \in B(L^2(G))^*$. In the general setting of locally compact quantum groups these descriptions are key to the development of the theory.

There are various procedures to construct finite quantum groups out of simpler ingredients. To conclude this section we shall briefly describe a prominent example, namely the *Drinfeld double* of a finite quantum group. For a proof of the following result, in a more general setting, see [30, Chapter 4].

Proposition 2.11. *Let G be a finite quantum group. Then*

$$C(\mathsf{D}(G)) = C(G) \otimes C^*(G),$$

equipped with the tensor product $*$-algebra structure, the comultiplication

$$\Delta_{\mathsf{D}(G)} = (\mathrm{id} \otimes \sigma \otimes \mathrm{id})(\mathrm{id} \otimes \mathsf{ad}(W) \otimes \mathrm{id})(\Delta \otimes \widehat{\Delta}),$$

the counit

$$\epsilon_{\mathsf{D}(G)}(f \otimes x) = \epsilon(f) \widehat{\epsilon}(x),$$

and the antipode

$$S_{\mathsf{D}(G)}(f \otimes x) = W^{-1}(S(f) \otimes \widehat{S}(x)) W$$

is a Hopf $*$-algebra, defining a finite quantum group $\mathsf{D}(G)$ with Haar state given by

$$\phi_{\mathsf{D}(G)} = \phi \otimes \widehat{\phi}.$$

This quantum group is called the Drinfeld double of G.

If G is a classical abelian group then $C(\mathsf{D}(G)) = C(G) \otimes C^*(G)$ is equipped simply with the tensor product Hopf $*$-algebra structure. However, if G is nonabelian then $C(\mathsf{D}(G))$ is neither commutative nor cocommutative, thus providing a first example of a finite quantum group which does not correspond to a group, or a group dual.

As explained above, general theory yields the Hopf $*$-algebra $C^*(\mathsf{D}(G))$ which plays the role of the group C^*-algebra of $\mathsf{D}(G)$. In the literature, often the Hopf algebra $C^*(\mathsf{D}(G))$ is called the Drinfeld double $C(G)$, and $C(\mathsf{D}(G))$ is sometimes referred to as the Drinfeld codouble.

3. Hopf C^*-algebras and locally compact quantum groups

In this section we extend our considerations to general locally compact quantum groups. The theory is significatly more involved, however, and we will only sketch some basic definitions and key facts.

Our starting point is the following definition.

Definition 3.1. *A Hopf C^*-algebra is a C^*-algebra H together with a nondegenerate $*$-homomorphism $\Delta : H \to M(H \otimes H)$ satisfying the coassociativity relation*

$$(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$$

and the density conditions

$$[\Delta(H)(H \otimes 1)] = H \otimes H = [\Delta(H)(1 \otimes H)].$$

Here $\otimes$ denotes the minimal tensor product of C^*-algebras.
A morphism between Hopf C^*-algebras (H, Δ_H) and (K, Δ_K) is a nondegenerate $*$-homomorphism $f : H \to K$ such that $\Delta_K f = (f \otimes f)\Delta_H$.

Comparing Definition 3.1 with Definition 2.1, it is conspicuous that there is no mentioning of a counit or antipode in the C^*-setting. It turns out that both these maps are typically unbounded in the operator algebraic setting, and therefore it is difficult to formalise them abstractly. However, we will see in section 6 how to construct ϵ and S for any *unital* Hopf C^*-algebra from the density conditions.

In the commutative case, the notion of a Hopf C^*-algebra has a very transparent interpretation. Recall that a semigroup G has *right cancellation* if $rt = st$ for all $t \in G$ implies $r = s$, and G has *left cancellation* if $tr = ts$ for all $t \in G$ implies $r = s$. We say that G has *cancellation* if it has both left and right cancellation.

Proposition 3.2. *Let H be a Hopf C^*-algebra whose underlying C^*-algebra is commutative. Then $H \cong C_0(G)$ for a locally compact semigroup G with cancellation. Conversely, every locally compact semigroup G with cancellation gives rise to a Hopf C^*-algebra $H = C_0(G)$.*

Proof. From the fact that H is commutative we get that $H = C_0(G)$ for a locally compact space G. Moreover, since $C_0(G) \otimes C_0(G) \cong C_0(G \times G)$, the nondegenerate $*$-homomorphism Δ induces a continuous map $G \times G \to G$, and coassociativity means that this turns G into a semigroup.

It remains to show that the cancellation conditions are equivalent to the density conditions. Assume first that the density condition $[\Delta(H)(1 \otimes H)] = H \otimes H$ holds. Morever assume that $r, s \in G$ satisfy $rt = st$ for all $t \in G$. Then $f_1(tx)f_2(x) = f_1(sx)f_2(x)$ for all $f_1, f_2 \in C_0(G)$, and since $\Delta(f_1)(1 \otimes f_2)(r, x) = f_1(rx)f_2(x)$ we see from the density condition that $f(s) = f(t)$ for all $f \in C_0(G)$, and this means $s = t$. Hence G satisfies right cancellation. In a similar way one shows that $[\Delta(H)(H \otimes 1)] = H \otimes H$ implies that G has left cancellation.

Conversely, assume that G has right cancellation and consider $A = \Delta(H)(1 \otimes H)$. Due to commutativity of H, it is easy to check that this is a $*$-subalgebra of $H \otimes H$. We claim that A separates the points of $G \times G$ and vanishes nowhere. Explicitly, for distinct points $(r_1, s_1), (r_2, s_2) \in G \times G$ we find an element of A of the form $h = \Delta(f)(1 \otimes g)$ such that $h(r_1, s_1) \neq h(r_2, s_2)$. If $s_1 \neq s_2$ it suffices to take $f, g \in C_0(G)$ such that $f(rs_1) = f(rs_2)$ and $g(s_1) \neq g(s_2)$, and if $s_1 = s = s_2$ we have $r_1 s \neq r_2 s$ by right cancellation, and we find $f, g \in C_0(G)$ such that $f(r_1 s) \neq f(r_2 s)$ and $g(s) \neq 0$. Moreover, for any $(r, s) \in G \times G$ we find $f, g \in C_0(G)$ such that $f(rs) \neq 0$ and $g(s) \neq 0$. Then $h = \Delta(f)(1 \otimes g) \in A$ satisfies $h(r, s) \neq 0$. The

Stone-Weierstraß theorem therefore implies that A is dense in $H \otimes H = C_0(G \times G)$. The proof of the other density condition is analogous. □

Proposition 3.2 shows that the notion of a Hopf C^*-algebra is not sufficient to encode the concept of a locally compact group. In order to obtain a sensible notion of a locally compact quantum group one needs to find additional conditions that single out groups from semigroups with cancellation. The most elegant solution to this problem, due Kustermans and Vaes [13], can be phrased as follows.

Definition 3.3. *A locally compact quantum group G is given by a Hopf C^*-algebra $H = C_0^{\mathrm{r}}(G)$ together with a faithful left Haar weight φ and a faithful right Haar weight ψ on $C_0^{\mathrm{r}}(G)$.*

Of course, the first task here is to explain what we mean by faithful left or right Haar weights. Rougly speaking, these weights are noncommutative versions of the left and right Haar measures of a locally compact group.

In order to make this precise let us review some related definitions and facts. Assume that A is a C^*-algebra and write A_+ for the positive part of A, so that

$$A_+ = \{a^*a \mid a \in A\}.$$

A *weight* on A is a map $\varphi : A_+ \to [0, \infty]$ such that

$$\varphi(a + b) = \varphi(a) + \varphi(b)$$
$$\varphi(\lambda a) = \lambda\varphi(a)$$

for all $a, b \in A_+$ and $\lambda \geq 0$. Here we use the convention $0 \cdot \infty = 0$.

A weight φ on A is called *lower semicontinuous* if for every $\lambda \geq 0$ the set $\{a \in A_+ \mid \varphi(a) \leq \lambda\}$ is closed in A. It is called *densely defined* if the set

$$\mathcal{N}_\varphi = \{a \in A \mid \varphi(a^*a) < \infty\}$$

is dense in A, and *faithful* if $\varphi(a^*a) = 0$ for $a \in A$ implies $a = 0$.

Definition 3.4. *Let A be a C^*-algebra and let $\varphi : A_+ \to [0, \infty]$ be a lower semicontinuous, densely defined weight. Assume that there exists a norm-continuous one-parameter group $(\sigma_t)_{t\in\mathbb{R}}$ of $*$-automorphisms of A such that $\varphi\sigma_t = \varphi$ for all $t \in \mathbb{R}$. Moreover assume that for all $a, b \in \mathcal{N}_\varphi \cap \mathcal{N}_\varphi^*$ there exists a bounded continuous function f on the strip $\{z \in \mathbb{C} \mid 0 \leq \Im(z) \leq 1\}$, analytic in the interior, satisfying*

$$f(t) = \varphi(\sigma_t(a)b), \qquad f(t + i) = \varphi(b\sigma_t(a))$$

for all $t \in \mathbb{R}$. Then φ is called a KMS-weight on A, and $(\sigma_t)_{t\in\mathbb{R}}$ is called a modular automorphism group of φ.

We remark that the modular automorphism group is uniquely determined by the weight φ if φ is faithful. Roughly speaking, the KMS-condition ensures that the weight behaves well when one wants to pass from the C^*-algebra level to the von Neumann algebra level.

A weight φ on a Hopf $*$-algebra H is called *left invariant* if for all positive linear functionals ω on A and $a \in A_+$ we have

$$\varphi((\omega \otimes \mathrm{id})\Delta(a)) = \varphi(a)\omega(1).$$

Here we note that $(\omega\otimes\mathrm{id})\Delta(a) \in A$, by the density condition $(A\otimes 1)\Delta(A) \subset A\otimes A$, and we recall that ω can be extended to a positive linear functional on the multiplier algebra $M(A)$, so that the expression $\omega(1)$ makes sense. In a similar way one defines the notion of right invariance.

Definition 3.5. *A left Haar weight on a Hopf C^*-algebra H is a left invariant, lower semicontinuous, densely defined KMS-weight on H. Similarly, a right Haar weight on H is a right invariant, lower semicontinuous, densely defined KMS-weight.*

As already indicated above, the following fundamental example motivates the terminology in Definition 3.5.

Example 3.6. *Let G be a locally compact group. Then the algebra $H = C_0(G)$ of continuous functions on G vanishing at infinity becomes a locally compact quantum group with the comultiplication $\Delta : C_0(G) \to M(C_0(G) \otimes C_0(G)) = C_b(G \times G)$ given by*

$$\Delta(f)(s,t) = f(st).$$

The left/right Haar weights on H are given by integration with respect to left/right Haar measure.

Let G be a locally compact quantum group, and let $\mathcal{H} = L^2(G)$ be a GNS-construction for the left Haar weight φ of G. We consider

$$\mathcal{N}_\varphi = \{f \in C_0^{\mathsf{r}}(G) \mid \varphi(f^* f) < \infty\},$$

and let $\Lambda : \mathcal{N}_\phi \to \mathcal{H}$ be the GNS-map. It is a nontrivial fact that one obtains a unitary operator $W \in B(\mathcal{H} \otimes \mathcal{H})$ by defining

$$W^*(\Lambda(f) \otimes \Lambda(g)) = (\Lambda \otimes \Lambda)(\Delta(g)(f \otimes 1))$$

for $f, g \in \mathcal{N}_\phi$, which we call the Kac-Takesaki operator of G. As in the case of finite quantum groups, the Kac-Takesaki operator W is multiplicative unitary, which means that it satisfies the *pentagon equation*

$$W_{12} W_{13} W_{23} = W_{23} W_{12}.$$

The GNS-representation of $C_0^{\mathsf{r}}(G)$ on $\mathcal{H}$ is faithful, so that one can view $C_0^{\mathsf{r}}(G)$ as a C^*-subalgebra of $B(\mathcal{H})$, and the Hopf C^*-algebra structure of $C_0^{\mathsf{r}}(G)$ can be described completely in terms of W by

$$C_0^{\mathsf{r}}(G) = [(\mathrm{id} \otimes B(\mathcal{H})_*)(W)], \qquad \Delta(f) = W^*(1 \otimes f)W.$$

Here $B(\mathcal{H})_*$ denotes the predual of $B(\mathcal{H})$, consisting or all σ-weakly continuous bounded linear functionals on $B(\mathcal{H})$. Moreover, one obtains a second Hopf C^*-algebra $C^*_{\mathsf{r}}(G)$ by setting

$$C^*_{\mathsf{r}}(G) = [(B(\mathcal{H})_* \otimes \mathrm{id})(W)], \qquad \hat{\Delta}(x) = \widehat{W}^*(1 \otimes x)\widehat{W}.$$

This generalizes the construction of the reduced group C^*-algebra of a locally compact group. The Hopf C^*-algebra $C^*_{\mathsf{r}}(G)$ admits again faithful left and right Haar weights, thus defining a locally compact quantum group $\hat{G}$, called the dual of G. It can be shown that $W \in M(C_0^{\mathsf{r}}(G) \otimes C^*_{\mathsf{r}}(G)) \subset B(\mathcal{H} \otimes \mathcal{H})$. For the details we refer to [13].

There are also maximal versions $C_0^{\mathsf{f}}(G)$ and $C^*_{\mathsf{f}}(G)$ of the function algebra and the group algebra of a locally compact quantum group [12]. These are again Hopf C^*-algebras, and there are canonical surjective $*$-homomorphisms $C_0^{\mathsf{f}}(G) \to C_0^{\mathsf{r}}(G)$ and $C^*_{\mathsf{f}}(G) \to C^*_{\mathsf{r}}(G)$, which are morphisms of Hopf C^*-algebras. Roughly speaking, the relation between $C_0^{\mathsf{f}}(G)$ and $C_0^{\mathsf{r}}(G)$ and the relation between $C^*_{\mathsf{f}}(G)$ and $C^*_{\mathsf{r}}(G)$ is like the relation between the full and reduced group C^*-algebras of a locally compact group.

Definition 3.7. *A compact quantum group is a locally compact quantum group G for which the C^*-algebra $C_0^{\mathsf{f}}(G)$ is unital.*

If G is a compact quantum we will write $C^{\mathsf{f}}(G) = C_0^{\mathsf{f}}(G)$ and $C^{\mathsf{r}}(G) = C_0^{\mathsf{r}}(G)$. Note that both $C^{\mathsf{f}}(G)$ and $C^{\mathsf{r}}(G)$ are unital Hopf C^*-algebras.

4. Compact quantum groups

We shall now focus our attention on the theory of compact quantum groups. Instead of working directly from Definition 3.7, we will however start from a weaker hypothesis, namely by studying arbitrary unital Hopf C^*-algebras. This has various advantages, and makes it easier to describe concrete examples since less data is required. It is a nontrivial fact that this gives essentially the same result as the theory of compact quantum groups in the sense of Definition 3.7, and we will only come back to this point at the end of section 6.

The most important result about unital Hopf C^*-algebras, which really makes everything work, is the following theorem due to Woronowicz [33].

Theorem 4.1. *Let H be a unital Hopf C^*-algebra. Then there exists a unique state ϕ on H such that*

$$(\phi \otimes \mathrm{id})\Delta(f) = \phi(f)1 = (\mathrm{id} \otimes \phi)\Delta(f)$$

for all $f \in H$.

For a proof of Theorem 4.1 see for instance [23, Section 1.2]. We note that uniqueness of ϕ is easy; indeed, the same argument as for Exercise 4 in section 2 works.

In view of Theorem 4.1, the difference between a general unital Hopf C^*-algebra H and a compact quantum group in the sense of Definition 3.7 is that the Haar state $\phi : H \to \mathbb{C}$ need not be faithful. Also, it is not clear at this point that ϕ satisfies the KMS-property.

Exercise 8. *Show that every compact semigroup with cancellation is a compact group. Use this to show that commutative unital Hopf C^*-algebras correspond to compact groups.*

We also obtain examples of unital Hopf C^*-algebras starting from arbitrary discrete groups.

Example 4.2. *Let Γ be a discrete group. Show that the full group C^*-algebra $C^*_{\mathsf{f}}(\Gamma)$ is a unital Hopf C^*-algebra with coproduct $\widehat{\Delta}(u_t) = u_t \otimes u_t$ for all $t \in \Gamma$. Moreover show that the same formula defines a unital Hopf C^*-algebra structure on $C^*_{\mathsf{r}}(\Gamma)$.*

Perhaps the most prominent example in the theory is the compact quantum group $SU_q(2)$ of Woronowicz [32].

Definition 4.3. *Let $q \in (0, 1]$. The C^*-algebra $C(SU_q(2))$ is the universal C^*-algebra generated by elements α and γ satisfying the relations*

$$\alpha\gamma = q\gamma\alpha, \quad \alpha\gamma^* = q\gamma^*\alpha, \quad \gamma\gamma^* = \gamma^*\gamma, \quad \alpha^*\alpha + \gamma^*\gamma = 1, \quad \alpha\alpha^* + q^2\gamma\gamma^* = 1.$$

The comultiplication $\Delta : C(SU_q(2)) \to C(SU_q(2)) \otimes C(SU_q(2))$ is defined by

$$\Delta(\alpha) = \alpha \otimes \alpha - q\gamma^* \otimes \gamma, \qquad \Delta(\gamma) = \gamma \otimes \alpha + \alpha^* \otimes \gamma.$$

Since the entries of a unitary matrix have norm at most one, the C^*-algebra $C(SU_q(2))$ can be viewed as the completion of the $*$-algebra generated by α and γ, with the relations in 4.3 and the norm obtained by considering arbitrary $*$-representations on a Hilbert space. From universality of $C(SU_q(2))$ one checks easily that $\Delta : C(SU_q(2)) \to C(SU_q(2)) \otimes C(SU_q(2))$ is indeed a well-defined unital $*$-homomorphism.

To verify that $C(SU_q(2))$ is a unital Hopf C^*-algebra we need the following lemma, compare [23, Proposition 1.1.4].

Lemma 4.4. *Assume that H is a unital C^*-algebra generated by elements u_{ij} for $1 \leq i, j \leq n$ such that the matrix $u = (u_{ij})$ and $\overline{u}$, the transpose of u^*, are invertible, and that $\Delta : H \to H \otimes H$ is a unital $*$-homomorphism such that*

$$\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$$

for all i, j. Then H is a unital Hopf C^*-algebra.

Proof. It is evident that Δ satisfies coassociativity, and it only remains to check the density conditions. To prove that $\Delta(H)(1 \otimes H)$ is dense in $H \otimes H$ it suffices to show that the space

$$B = \{f \in H \mid f \otimes 1 = \sum_i \Delta(a_i)(1 \otimes b_i) \text{ for some } a_i, b_i \in H\}$$

is dense in H. Here ths sum in the expression on the right hand side is finite.

Note first that B is an algebra since if $f = \sum_i \Delta(a_i)(1 \otimes b_i), g = \sum_j \Delta(x_j)(1 \otimes y_j)$ then

$$\begin{aligned} fg \otimes 1 &= \sum_i \Delta(a_i)(1 \otimes b_i)(g \otimes 1) \\ &= \sum_i \Delta(a_i)((g \otimes 1)(1 \otimes b_i) \\ &= \sum_{i,j} \Delta(a_i x_i)(1 \otimes y_j b_i). \end{aligned}$$

Hence it is enough to show that B contains the generators u_{ij} and their adjoints. To this end let (v_{ij}) be the inverse of u and write

$$u_{ij} \otimes 1 = \sum_k u_{ik} \otimes \delta_{kj} = \sum_{k,l} u_{ik} \otimes u_{kl} v_{lj} = \sum_l \Delta(u_{il})(1 \otimes v_{lj}.$$

Similarly let (w_{ij}) be the inverse of $\overline{u}$ and compute

$$u^*_{ij} \otimes 1 = \sum_k u^*_{ik} \otimes \delta_{kj} = \sum_{k,l} u^*_{ik} \otimes u^*_{kl} w_{lj} = \sum_l \Delta(u^*_{il})(1 \otimes w_{lj}.$$

We conclude that B is dense in H, and hence $[\Delta(H)(1 \otimes H)] = H \otimes H$.

The other density condition $[\Delta(H)(H \otimes 1)] = H \otimes H$ is verified in a similar way. □

Now let $H = C(SU_q(2))$ for some $q \in (0, 1]$ and consider

$$u = \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix} = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \in M_2(C(SU_q(2))).$$

It is easy to check that the defining relations in Definition 4.3 are equivalent to saying that u is a unitary matrix, and we get

$$\begin{aligned} \overline{u} = \begin{pmatrix} \alpha^* & -q\gamma \\ \gamma^* & \alpha \end{pmatrix} &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha & -\gamma^* \\ q\gamma & \alpha^* \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} q^{-1/2} & 0 \\ 0 & q^{1/2} \end{pmatrix} \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix} \begin{pmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -q^{1/2} \\ q^{-1/2} & 0 \end{pmatrix} u \begin{pmatrix} 0 & q^{1/2} \\ -q^{-1/2} & 0 \end{pmatrix}. \end{aligned}$$

Hence both u and $\overline{u}$ are invertible, and we can apply Lemma 4.4 to see that $C(SU_q(2))$ is indeed a unital Hopf C^*-algebra.

Exercise 9. *Show that for $q = 1$ the definition of $SU_q(2)$ recovers the classical group $SU(2)$, that is, show that there is an isomorphism*

$$C(SU_1(2)) \cong C(SU(2))$$

of Hopf C^*-algebras.

Another source of examples comes from "quantum" analogues of free groups, introduced by Wang and Van Daele in [28].

Definition 4.5. *Let $F \in GL(n, \mathbb{C})$ such that $\mathrm{tr}(F^*F) = \mathrm{tr}((F^*F)^{-1})$. The free unitary quantum group U_F^+ is defined by the universal C^*-algebra $A_u(F) = C(U_F^+)$ generated by elements u_{ij} for $1 \leq i, j \leq n$ satisfying the relations such that*

$$u = (u_{ij}), \qquad F\overline{u}F^{-1}$$

are unitary. The comultiplication is given by $\Delta(u_{ij}) = \sum_{k=1}^n u_{ik} \otimes u_{kj}$.

The origin of the matrix F in Definition 4.6 will become clear in our discussion of the representation theory of compact quantum groups in section 6.

If $F = \mathrm{id}$ is the identity matrix we also write U_n^+ for U_F^+, and $A_u(n)$ instead of $A_u(F)$. The definining relations in Definition 4.6 imply that there is a canonical surjective $*$-homomorphism $C(U_n^+) \to C(U_n)$ where U_n is the group of unitary $n \times n$ matrices.

Exercise 10. *Show that the C^*-algebra $C(U_n)$ of functions on the unitary group U_n is the abelianisation of $C(U_n^+)$.*

One may write $A_u(F) = C^*(\mathbb{F}U_F)$ and think of this as an analogue of the full group C^*-algebra of a free group. Indeed, there are various results about the structure of these quantum groups supporting this point of view.

Let us next introduce an analogue of the above construction for orthogonal groups, also discussed in [28].

Definition 4.6. *Let $F \in GL(n, \mathbb{C})$ such that $F\overline{F} = \pm 1$. The free orthogonal quantum group O_F^+ is defined by the universal C^*-algebra $A_o(F) = C(O_F^+)$ generated by elements u_{ij} for $1 \leq i, j \leq n$ satisfying the relations such that $u = (u_{ij})$ is unitary and*

$$F\overline{u}F^{-1} = u.$$

The comultiplication is again given by $\Delta(u_{ij}) = \sum_{k=1}^n u_{ik} \otimes u_{kj}$.

We note that $F\overline{F} = \pm 1$ implies that $\mathrm{tr}(F^*F) = \mathrm{tr}((F^*F)^{-1})$. Therefore $A_o(F)$ is a quotient of $A_u(F)$. If $F = \mathrm{id}$ is the identity matrix we write O_n^+ for O_F^+ and $A_o(n)$ for $A_o(F)$, and it is not difficult to check that there is a canonical surjective $*$-homomorphism $C(O_n^+) \to C(O_n)$, where O_n is the group of orthogonal $n \times n$ matrices, identifying $C(O_n)$ with the abelianisation of $C(O_n^+)$.

To conclude this section, let us discuss the quantum permutation groups introduced by Wang [31].

Definition 4.7. *Let $n \in \mathbb{N}$. The quantum permutation group S_n^+ is the compact quantum group given by the universal C^*-algebra generated by the entries of a magic unitary $n \times n$-matrix $u = (u_{ij})$, that is, $C(S_n^+) = A_s(n)$ is the universal C^*-algebra generated by projections u_{ij} for $1 \leq i, j \leq n$ such that*

$$\sum_{i=1}^n u_{ik} = 1, \qquad \sum_{j=1}^n u_{kj} = 1$$

for all $1 \leq k \leq n$. The comultiplication $\Delta : C(S_n^+) \to C(S_n^+) \otimes C(S_n^+)$ is defined by $\Delta(u_{ij}) = \sum_{k=1}^n u_{ik} \otimes u_{kj}$ on the generators.

One obtains a canonical morphism $C(S_n^+) \to C(S_n)$ of Hopf C^*-algebras, where S_n is the symmetric group on n elements. In fact, in analogy to the case of $A_u(n)$ and $A_o(n)$ one checks that $C(S_n)$ is the abelianisation of $C(S_n^+)$.

The structure of the quantum permutation group S_n^+ easy to understand for $n = 1, 2, 3$.

Exercise 11. *Show that the canonical morphism of Hopf C^*-algebras $C(S_n^+) \to C(S_n)$ is an isomorphism for $n = 1, 2, 3$.*

For $n \geq 4$ the C^*-algebra $C(S_n^+)$ is infinite dimensional, and the morphism $C(S_n^+) \to C(S_n)$ is no longer an isomorphism. In other words, for $n \geq 4$ there are genuine "quantum permutations" which are not visible from a classical perspective. There are interesting connections with free probability, the theory of nonlocal games, and graph theory which all build on this fact [2], [16], [20].

5. Representation theory

Let us now discuss some elements of the representation theory of locally compact quantum groups. For more background we refer to [33], [23], [18].

Definition 5.1. *Let G be a locally compact quantum group. A unitary representation $\pi = (\mathcal{H}_\pi, U^\pi)$ of G consists of a Hilbert space $\mathcal{H}_\pi$ and a unitary element $U^\pi \in M(C_0^{\mathrm{r}}(G) \otimes K(\mathcal{H}_\pi))$ such that*

$$(\Delta \otimes \mathrm{id})(U^\pi) = U^\pi_{13} U^\pi_{23}$$

in $M(C_0^{\mathrm{r}}(G) \otimes C_0^{\mathrm{r}}(G) \otimes K(\mathcal{H}_\pi))$.

In this definition we are using the leg numbering notation from Section 2. Specifically, if $X \in M(C_0^{\mathrm{r}}(G) \otimes K(\mathcal{H}))$ we write $X_{23} = 1 \otimes X$ for the element X placed in the last two tensor legs, and similarly X_{13} is acting in the first and third leg.

We note that Definition 5.1 makes sense for arbitrary Hopf C^*-algebras. In this case, a unitary element $U \in M(H \otimes K(\mathcal{H}))$ satisfying $(\Delta \otimes \mathrm{id})(U) = U_{13}U_{23}$ is also called a *unitary corepresentation* of H on $\mathcal{H}$.

Example 5.2. *The* trivial *representation of a locally compact quantum group G is the unitary representation ϵ on $\mathcal{H}_\epsilon = \mathbb{C}$ given by $U^\epsilon = 1 \in M(C_0^{\mathrm{r}}(G)) \cong M(C_0^{\mathrm{r}}(G) \otimes \mathbb{C})$.*

We should check that Definition 5.1 is compatible with the standard definition of unitary representations for classical locally compact groups.

Recall that if G is a locally compact group, then a unitary representation π of G on a Hilbert space $\mathcal{H}_\pi$ in the classical sense is a strongly continuous group homomorphism $\pi : G \to U(\mathcal{H}_\pi)$, where $U(\mathcal{H}_\pi)$ is the unitary group of $\mathcal{H}_\pi$, see for instance [10, Chapter 3]. That is, the group homomorphism π is required to be continuous with respect to the given topology of G and the strong operator topology on $U(\mathcal{H}_\pi) \subset B(\mathcal{H}_\pi)$.

Proposition 5.3. *Let G be a locally compact group. Then the definition of a unitary representation of G in the sense of locally compact quantum groups is equivalent to the classical definition of a unitary representation of G.*

Proof. Let $\mathcal{H}$ be a Hilbert space. Then the multiplier algebra $M(C_0(G) \otimes K(\mathcal{H}))$ identifies canonically with the algebra $C_b(G, B(\mathcal{H}))$ of strictly continuous bounded functions $G \to B(\mathcal{H})$, compare [25, Proposition 2.57]. We also recall that the strict topology of $B(\mathcal{H}) = M(K(\mathcal{H}))$ agrees with the strong*-topology on bounded sets [25, Lemma C.6].

It follows from these facts that a unitary element $X \in M(C_0(G) \otimes K(\mathcal{H}))$ is the same thing as a strongly continuous map $G \to U(\mathcal{H})$ into the unitary group of $\mathcal{H}$, which we will again denote by X. Moreover we have

$$(\Delta \otimes \mathrm{id})(X)(s,t) = X(st)$$

and

$$(X_{13}X_{23})(s,t) = X(s)X(t)$$

for all s,t, interpreting both sides as functions on $G \times G$ with values in $B(\mathcal{H})$. Therefore a unitary $X \in M(C_0(G) \otimes K(\mathcal{H}))$ defines a unitary representation $X : G \to U(\mathcal{H})$ in the classical sense iff $(\Delta \otimes \mathrm{id})(X) = X_{13}X_{23}$. □

For any locally compact quantum group G, an important example of a unitary representation is the *left regular representation*. By definition, the left regular representation $\lambda = (L^2(G), W)$ is the GNS-space $L^2(G)$ of the left Haar weight together with the Kac-Takesaki operator $W \in M(C_0^{\mathrm{r}}(G) \otimes C_{\mathrm{r}}^*(G)) \subset M(C_0^{\mathrm{r}}(G) \otimes K(L^2(G)))$. Here we are using the natural embedding $C_{\mathrm{r}}^*(G) \to B(L^2(G))$, which by slight abuse of notation is usually also denoted by λ. The pentagon equation for W implies that $(\Delta \otimes \mathrm{id})(W) = W_{13}W_{23}$ as required.

Exercise 12. *Check that $(L^2(G), W)$ reproduces the left regular representation $\lambda : G \to U(L^2(G))$ given by*

$$\lambda_t(\xi)(s) = \xi(t^{-1}s)$$

for $\xi \in L^2(G)$ if G is a classical locally compact group.

Let us continue with some further definitions.

Definition 5.4. *Let G be a locally compact quantum group and let $\pi = (\mathcal{H}_\pi, U^\pi), \eta = (\mathcal{H}_\eta, U^\eta)$ be unitary representations of G. An* intertwiner *from π to η is a bounded linear operator $T \in B(\mathcal{H}_\pi, \mathcal{H}_\eta)$ such that*

$$(1 \otimes T)U^\pi = U^\eta(1 \otimes T).$$

We write $\mathrm{Mor}(\pi, \eta)$ for the space of all intertwiners from π to η. Note that if $T \in \mathrm{Mor}(\pi, \eta)$ then $T^* \in \mathrm{Mor}(\eta, \pi)$. In particular, this implies that the space $\mathrm{Mor}(\pi, \pi)$ of self-intertwiners of π is a unital C^*-algebra.

We say that π and η are *equivalent* if there exists a unitary intertwiner between them, and we write $\pi \cong \eta$ in this case. A unitary representation π is called *irreducible* if $\mathrm{Mor}(\pi, \pi) = \mathbb{C}\,\mathrm{id}$.

Lemma 5.5 (Schur's Lemma)**.** *Let π, η be irreducible unitary representations of a locally compact quantum group G. Then $\mathrm{Mor}(\pi, \eta) \cong \mathbb{C}$ iff π and η are equivalent, and $\mathrm{Mor}(\pi, \eta) = 0$ otherwise.*

Proof. Pick a nonzero element $T \in \mathrm{Mor}(\pi, \eta)$. Then $T^*T \in \mathrm{Mor}(\pi, \pi) = \mathbb{C}\,\mathrm{id}$, hence $T^*T = c\,\mathrm{id}$ for some scalar $c > 0$. Similarly $TT^* \in \mathrm{Mor}(\eta, \eta) = \mathbb{C}\,\mathrm{id}$ is a nonzero multiple of the identity. It follows that $c^{-1/2}T$ is a unitary intertwiner from π to η. Hence π and η are equivalent. If π and η are not equivalent this argument shows that $\mathrm{Mor}(\pi, \eta) = 0$. □

If $\pi = (\mathcal{H}_\pi, U^\pi)$ is a unitary representation of G and $\mathcal{K} \subset \mathcal{H}_\pi$ is a closed subspace such that the orthogonal projection p onto $\mathcal{K}$ is an element of $\mathrm{Mor}(\pi, \pi)$ then we say that $\mathcal{K}$ is a *subrepresentation*, or *invariant subspace*, of π. Note that in this case U^π induces canonically an element $U^{\pi_{|\mathcal{K}}} = (1 \otimes p)U^\pi = U^\pi(1 \otimes p) \in M(C_0^{\mathrm{r}}(G) \otimes K(\mathcal{K}))$, which defines a unitary representation $\pi_{|\mathcal{K}} = (\mathcal{K}, U^{\pi_{|\mathcal{K}}})$ in its own right.

If $\pi = (\mathcal{H}_\pi, U^\pi), \eta = (\mathcal{H}_\eta, U^\eta)$ are unitary representations of G then the *direct sum* $\pi \oplus \eta$ is the unitary representation on $\mathcal{H}_\pi \oplus \mathcal{H}_\eta$ defined by

$$U^{\pi \oplus \eta} = U^\pi \oplus U^\eta,$$

using $M(C_0^{\mathrm{r}}(G) \otimes K(\mathcal{H}_\pi)) \oplus M(C_0^{\mathrm{r}}(G) \otimes K(\mathcal{H}_\eta)) \subset M(C_0^{\mathrm{r}}(G) \otimes K(\mathcal{H}_\pi \oplus \mathcal{H}_\eta))$. In a similar way one defines infinite direct sums.

The *tensor product* $\pi \otimes \eta$ of unitary representations $\pi = (\mathcal{H}_\pi, U^\pi), \eta = (\mathcal{H}_\eta, U^\eta)$ of G is defined by $U^{\pi\otimes\eta} = U^\pi_{12} U^\eta_{13} \in M(C^r_0(G) \otimes K(\mathcal{H}_\pi \otimes \mathcal{H}_\eta))$. Using

$$\begin{aligned}(\Delta \otimes \mathrm{id})(U^{\pi\otimes\eta}) &= (\Delta \otimes \mathrm{id})(U^\pi_{12} U^\eta_{13}) \\ &= (\Delta \otimes \mathrm{id})(U^\pi_{12})(\Delta \otimes \mathrm{id})(U^\eta_{13}) \\ &= U^\pi_{13} U^\pi_{23} U^\eta_{14} U^\eta_{24} \\ &= U^\pi_{13} U^\eta_{14} U^\pi_{23} U^\eta_{24} \\ &= U^{\pi\otimes\eta}_{13} U^{\pi\otimes\eta}_{23}\end{aligned}$$

we see that $\pi \otimes \eta$ is a unitary representation of G on $\mathcal{H}_{\pi\otimes\eta} = \mathcal{H}_\pi \otimes \mathcal{H}_\eta$.

Exercise 13. *Let π, η, ρ be unitary representations of a locally compact quantum group G. Show that*

$$(\pi \otimes \eta) \otimes \rho \cong \pi \otimes (\eta \otimes \rho).$$

Moreover, show that $\pi \otimes \epsilon \cong \pi \cong \epsilon \otimes \pi$, where ϵ is the trivial representation.

The collection of all unitary representations of a locally compact quantum group forms naturally a C^**-tensor category*. Moreover, in the case of compact quantum groups, the C^*-tensor category of finite dimensional unitary representations is *rigid*. This allows one to extend various techniques from the study of discrete groups to the realm of compact quantum groups, and provides a link between quantum groups and the theory of subfactors. We refer to [23] for further information.

6. Representations of compact quantum groups

From now on we will focus our attention on the representation theory of *compact* quantum groups. In fact, we will rather consider arbitrary unital Hopf C^*-algebras and their corepresentations. By slight abuse of terminology, at this point, we will speak of a compact quantum group G in the sequel, meaning an arbitrary unital Hopf C^*-algebra $H = C(G)$. At the end of this section we explain how this links up with Definition 3.7.

Let us first construct the left regular representation of G on $L^2(G)$. Here $L^2(G)$ denotes the GNS-space of the Haar state ϕ, with associated GNS-map $\Lambda : C(G) \to L^2(G)$ and inner product $\langle \Lambda(f), \Lambda(g)\rangle = \phi(f^* g)$ for all $f, g \in C(G)$. In the sequel we will identify $M(C(G) \otimes K(L^2(G))$ with the algebra $B(C(G) \otimes L^2(G))$ of adjointable operators on the Hilbert $C(G)$-module $C(G) \otimes L^2(G)$, with the inner product given by $\langle f \otimes \xi, g \otimes \eta\rangle = f^* g\langle \xi, \eta\rangle$, compare [15, Theorem 2.4].

With this in mind, we define a linear operator $W^* \in B(C(G) \otimes L^2(G))$ by setting

$$W^*(f \otimes \Lambda(g)) = (\mathrm{id} \otimes \Lambda)(\Delta(g)(f \otimes 1)).$$

Then we obtain

$$\begin{aligned}\langle W^*(f \otimes \Lambda(g)), W^*(h \otimes \Lambda(k))\rangle &= (\mathrm{id} \otimes \phi)((f^* \otimes 1)\Delta(g^*)\Delta(k)(h \otimes 1)) \\ &= f^*(\mathrm{id} \otimes \phi)(\Delta(g^* k))h \\ &= f^* h\langle \Lambda(g), \Lambda(k)\rangle \\ &= \langle f \otimes \Lambda(g), h \otimes \Lambda(k)\rangle,\end{aligned}$$

which means that W^* is indeed a well-defined isometry. By the density condition for $H = C(G)$ we see that the image of W^* is dense in $C(G) \otimes L^2(G)$, and we conclude that W^* is a unitary.

Lemma 6.1. *The operator $W \in M(C(G) \otimes K(L^2(G)))$ defines a unitary representation of G on $L^2(G)$.*

Proof. It suffices to show $(\Delta \otimes \mathrm{id})(W^*) = W_{23}^* W_{13}^*$. Note first that since W^* is right $C(G)$-linear we can write

$$W^*(f \otimes \Lambda(h)) = (\mathrm{id} \otimes \widehat{\lambda})(\Delta(h))(1 \otimes \Lambda(1))f,$$

where $\widehat{\lambda} : C(G) \to B(L^2(G))$ is the GNS-representation. Using this description we get

$$\begin{aligned}(\Delta \otimes \mathrm{id})&(W^*)(f \otimes g \otimes \Lambda(h)) \\ &= (\mathrm{id} \otimes \mathrm{id} \otimes \widehat{\lambda})((\Delta \otimes \mathrm{id})\Delta(h))((\mathrm{id} \otimes \mathrm{id} \otimes \Lambda)(1 \otimes 1 \otimes 1))(f \otimes g) \\ &= (\mathrm{id} \otimes \mathrm{id} \otimes \Lambda)((\Delta \otimes \mathrm{id})\Delta(h)(f \otimes g \otimes \mathrm{id})),\end{aligned}$$

and then

$$\begin{aligned}(\Delta \otimes \mathrm{id})(W^*)(f \otimes g \otimes \Lambda(h)) &= (\mathrm{id} \otimes \mathrm{id} \otimes \Lambda)((\Delta \otimes \mathrm{id})\Delta(h)(f \otimes g \otimes \mathrm{id})) \\ &= (\mathrm{id} \otimes \mathrm{id} \otimes \Lambda)((\mathrm{id} \otimes \Delta)\Delta(h)(f \otimes g \otimes \mathrm{id})) \\ &= W_{23}^*(\mathrm{id} \otimes \mathrm{id} \otimes \Lambda)(\Delta(h)_{13}(f \otimes g \otimes 1)) \\ &= W_{23}^* W_{13}^*(f \otimes g \otimes \Lambda(h))\end{aligned}$$

for all $f, g, h \in C(G)$. Since elements of this form span a dense linear subspace of the Hilbert module $C(G) \otimes C(G) \otimes L^2(G)$ this yields the claim. □

Proposition 6.2. *The left regular representation $\lambda = (L^2(G), W)$ has the following properties.*

a) We have $[(\mathrm{id} \otimes \omega_{\xi,\eta})(W) \mid \xi, \eta \in L^2(G)] = C(G)$.

b) We have $W^*(1 \otimes \widehat{\lambda}(f))W = (\mathrm{id} \otimes \widehat{\lambda})\Delta(f)$ for all $f \in C(G)$.

Proof. *a)* Since $C(G)$ is a $*$-algebra it suffices to establish the equivalent equality $[(\mathrm{id} \otimes \omega_{\xi,\eta})(W^*) \mid \xi, \eta \in L^2(G)] = C(G)$. For $\xi = \Lambda(g), \eta = \Lambda(h)$ we compute

$$\begin{aligned}(\mathrm{id} \otimes \omega_{\xi,\eta})(W^*)(f) &= \langle 1 \otimes \Lambda(g), W^*(f \otimes \Lambda(h))\rangle \\ &= \langle (\mathrm{id} \otimes \Lambda)(1 \otimes g), (\mathrm{id} \otimes \Lambda)\Delta(h)(f \otimes 1)\rangle \\ &= (\mathrm{id} \otimes \phi)((g^* \otimes 1)\Delta(h))f\end{aligned}$$

for all $f \in C(G)$. Since $[(C(G) \otimes 1)\Delta(C(G))] = C(G) \otimes C(G)$ it follows that the operators $(\mathrm{id} \otimes \omega_{\xi,\eta})(W^*)$ yield indeed a dense linear subspace of $B(C(G)) = C(G)$.
b) We calculate

$$\begin{aligned}W^*(1 \otimes \widehat{\lambda}(f))(h \otimes \Lambda(g)) &= W^*(h \otimes \Lambda(fg)) \\ &= (\mathrm{id} \otimes \Lambda)(\Delta(fg)(h \otimes 1) \\ &= (\mathrm{id} \otimes \widehat{\lambda})\Delta(f)(\mathrm{id} \otimes \Lambda)(h \otimes \Lambda(g)) \\ &= (\mathrm{id} \otimes \widehat{\lambda})\Delta(f)W^*(h \otimes \Lambda(g))\end{aligned}$$

for all $g, h \in C(G)$. This yields the claim since W is unitary. □

We now come to the first main result concerning representations of compact quantum groups.

Theorem 6.3. *Every unitary representation of a compact quantum group is equivalent to a direct sum of irreducible representations, and every irreducible representation is finite dimensional.*

Proof. Assume that $\pi = (\mathcal{H}, U)$ is a unitary representation of the compact quantum group G. It suffices to show that $\mathrm{Mor}(\pi, \pi)$ contains a nonzero positive compact operator T. Then the spectral projection associated to a strictly positive eigenvalue of T gives a finite dimensional invariant subspaces of $\mathcal{H}$, and a Lemma of Zorn argument completes the proof, compare [10, Theorem 5.2].

In order to construct a compact self-intertwiner of π take an arbitrary element $L \in K(\mathcal{H})$. Then $U^*(1 \otimes L)U$ is contained in $C(G) \otimes K(\mathcal{H})$, and hence the operator

$$T = (\phi \otimes \mathrm{id})(U^*(1 \otimes L)U)$$

is contained in $K(\mathcal{H})$. Moreover we calculate

$$\begin{aligned} U^*(1 \otimes T)U &= (\phi \otimes \mathrm{id} \otimes \mathrm{id})(U_{23}^* U_{13}^*(1 \otimes 1 \otimes L)U_{13}U_{23}) \\ &= (\phi \otimes \mathrm{id} \otimes \mathrm{id})((\Delta \otimes \mathrm{id})(U^*)(1 \otimes 1 \otimes L)(\Delta \otimes \mathrm{id})(U)) \\ &= (\phi \otimes \mathrm{id} \otimes \mathrm{id})((\Delta \otimes \mathrm{id})(U^*(1 \otimes L)U) \\ &= 1 \otimes (\phi \otimes \mathrm{id})(U^*(1 \otimes L)U) \\ &= 1 \otimes T, \end{aligned}$$

using the invariance of the Haar state from Theorem 4.1. Equivalently, we have $U(1 \otimes T) = (1 \otimes T)U$, which means $T \in \mathrm{Mor}(\pi, \pi)$.

We note that it may well happen that $T = 0$ even though L is nonzero. However, if p_i is a net of finite rank projections in $B(\mathcal{H})$ converging weakly to the identity then the operators $T_i = (\phi \otimes \mathrm{id})(U^*(1 \otimes p_i)U)$ converge weakly to the identity as well. Explicitly, if $\xi, \eta \in \mathcal{H}$ then

$$\omega_{\xi,\eta}((\phi \otimes \mathrm{id})(U^*(1 \otimes p_i)U)) = \langle \Lambda(1) \otimes \xi, (U^*(1 \otimes p_i)U)(\Lambda(1) \otimes \eta)\rangle,$$

and the right hand side converges to $\langle \xi, \eta \rangle = \omega_{\xi,\eta}(\mathrm{id})$. It follows that $\mathrm{Mor}(\pi, \pi)$ contains indeed a nonzero positive compact operator as required.

Now assume that π is irreducible. Then the space $\mathrm{Mor}(\pi, \pi)$ contains only scalar multiples of the identity by Lemma 5.5. Hence the above discussions shows that the identity operator on $\mathcal{H}$ is compact, which means that the Hilbert space $\mathcal{H}$ must be finite dimensional. □

Due to Theorem 6.3 it suffices to focus our attention on (irreducible) finite dimensional representations.

Proposition 6.4. *Let $\pi = (\mathcal{H}, U)$ be a finite dimensional unitary representation of the compact quantum group G. Then*

$$B = [(\phi \otimes \mathrm{id})(U(g \otimes \mathrm{id})) \mid g \in C(G)]$$

is a unital C^*-algebra $B \subset B(\mathcal{H})$ such that $U \in C(G) \otimes B$. Moreover, if π is irreducible then $B = B(\mathcal{H})$.

Proof. For $g \in C(G)$ we define

$$\lambda(g) = (\phi \otimes \mathrm{id})(U(g \otimes \mathrm{id})).$$

Then B is equal to the set of all $\lambda(g)$ for $g \in C(G)$. Indeed, these elements form a linear subspace of B, and since $B(\mathcal{H})$ is finite dimensional, this subspace is automatically closed.

From the relation $(\Delta \otimes \mathrm{id})(U) = U_{13}U_{23}$ we get

$$U_{13}^*(\Delta \otimes \mathrm{id})(U(g \otimes \mathrm{id})) = U_{23}(\Delta(g) \otimes \mathrm{id}).$$

Applying $\mathrm{id} \otimes \phi \otimes \mathrm{id}$ on both sides gives

$$U^*(1 \otimes \lambda(g)) = (\mathrm{id} \otimes \phi \otimes \mathrm{id})(U_{23}\Delta(g) \otimes \mathrm{id}).$$

Multiplying this relation on the left by $f^* \otimes \mathrm{id}$ and applying $\phi \otimes \mathrm{id}$ gives

$$\begin{aligned} \lambda(f)^*\lambda(g) &= (\phi \otimes \mathrm{id})((f^* \otimes \mathrm{id})U^*(1 \otimes \lambda(g))) \\ &= (\phi \otimes \phi \otimes \mathrm{id})(U_{23}((f^* \otimes 1)\Delta(g)) \otimes \mathrm{id}). \end{aligned}$$

By the density conditions in Definition 3.1, elements of the form $(f^* \otimes 1)\Delta(g)$ span a dense linear subspace of $C(G) \otimes C(G)$. It follows that products of the form

$\lambda(f)^*\lambda(g)$ linearly span B. We conclude that $B = B^*$ and $BB = B$, which means that B is a C^*-algebra.

Since U is unitary we get $U(C(G) \otimes K(\mathcal{H})) = C(G) \otimes K(\mathcal{H})$, and this implies $BK(\mathcal{H}) = K(\mathcal{H})$. In particular B contains $\mathrm{id} \in K(\mathcal{H})$.

Finally, picking $h \in C(G)$ such that $\lambda(h) = \mathrm{id}$ we get from a relation established further above that

$$U^* = U^*(\mathrm{id} \otimes \lambda(h)) = (\mathrm{id} \otimes \phi \otimes \mathrm{id})(U_{23}\Delta(h) \otimes \mathrm{id}),$$

which means that U^* is contained in $C(G) \otimes B$. Hence the same holds for U.

It follows that every operator in the commutant of $B \subset B(\mathcal{H})$ is an intertwiner of π. If π is irreducible then this means $B = B(\mathcal{H})$. □

An n-dimensional unitary representation $\pi = (\mathcal{H}, U)$ of G can be viewed as a matrix $U = (U_{ij}) \in C(G) \otimes K(\mathcal{H}) \cong M_n(C(G))$ upon choosing an orthonormal basis $e_1, \dots, e_n$ of $\mathcal{H}$. More precisely, we may write

$$U = \sum_{i,j} U_{ij} \otimes e_{ij}$$

where $e_{ij} = |e_i\rangle\langle e_j|$ are the corresponding matrix units.

Lemma 6.5. *Let G be a compact quantum group, let $\mathcal{H}$ be a finite dimensional Hilbert space and consider $U = \sum_{i,j} U_{ij} \otimes e_{ij} \in C(G) \otimes K(\mathcal{H})$ with respect to some orthonormal basis $e_1, \dots, e_n$ of $\mathcal{H}$. Then the following conditions are equivalent.*

a) U defines a unitary representation of G on $\mathcal{H}$.
b) (U_{ij}) is a unitary matrix in $M_n(C(G))$ such that $\Delta(U_{ij}) = \sum_k U_{ik} \otimes U_{kj}$ for all $1 \leq i, j \leq n$.

If these equivalent conditions are satisfied, then an element $T = (T_{ij}) \in B(\mathcal{H})$ is an intertwiner of $(\mathcal{H}, U)$ iff

$$\sum_k T_{ik} U_{kj} = \sum_k U_{ik} T_{kj}$$

for all $1 \leq i, j \leq n$.

Proof. Clearly, the element U is unitary iff the matrix (U_{ij}) is unitary. Moreover we have

$$(\Delta \otimes \mathrm{id})(U) = \sum_{i,j} \Delta(U_{ij}) \otimes e_{ij}$$

and

$$U_{13}U_{23} = \sum_{i,k,l,j} U_{ik} \otimes U_{lj} \otimes e_{ik}e_{lj} = \sum_{i,k,j} U_{ik} \otimes U_{kj} \otimes e_{ij}.$$

Comparing these expressions, and using that the matrix units e_{ij} are linearly independent, yields the equivalence of a) and b).

Now consider $T \in B(\mathcal{H})$. Then we compute

$$(1 \otimes T)U = \sum_{i,j,k,l} (T_{ik} \otimes e_{ik})(U_{lj} \otimes e_{lj}) = \sum_{i,j,k} T_{ik}U_{kj} \otimes e_{ij}$$

and

$$U(1 \otimes T) = \sum_{i,j,k,l} (U_{ik} \otimes e_{ik})(T_{lj} \otimes e_{lj}) = \sum_{i,j,k} U_{ik}T_{kj} \otimes e_{ij}.$$

It follows that T is an intertwiner if and only if $T = (T_{ij})$, when viewed as a matrix in $C(G)$ with entries being scalar multiples of the identity, commutes with U. □

Let $\pi = (\mathcal{H}, U)$ be a finite dimensional unitary representation of a compact quantum group G and consider the map $\mathsf{ad}_l : K(\mathcal{H}) \to C(G) \otimes K(\mathcal{H})$ given by

$$\mathsf{ad}_l(T) = U^*(1 \otimes T)U.$$

Then ad_l is an injective unital $*$-homomorphism, and we calculate

$$\begin{aligned}(\Delta \otimes \mathrm{id})\mathsf{ad}_l(T) &= (\Delta \otimes \mathrm{id})(U^*(1 \otimes T)U) \\ &= (\Delta \otimes \mathrm{id})(U^*)(1 \otimes 1 \otimes T)(\Delta \otimes \mathrm{id})(U) \\ &= U_{23}^* U_{13}^*(1 \otimes 1 \otimes T)U_{13}U_{23} \\ &= (\mathrm{id} \otimes \mathsf{ad}_l)\mathsf{ad}_l(T).\end{aligned}$$

We call ad_l the *left adjoint action* on $K(\mathcal{H})$, and say that a state θ on $K(\mathcal{H})$ is *invariant* under this action if

$$(\mathrm{id} \otimes \theta)\mathsf{ad}_l(T) = \theta(T)1$$

for all $T \in K(\mathcal{H})$. In section 7 we will see that the left adjoint action is indeed an example of an action in the sense of quantum groups.

Theorem 6.6. *Assume that the unitary representation $\pi = (\mathcal{H}, U)$ is irreducible. Then $K(\mathcal{H})$ admits a unique invariant state with respect to the left adjoint action, and this state is faithful.*

Proof. To show existence, let ω be any state on $K(\mathcal{H})$ and consider $\theta = (\phi \otimes \omega)\mathsf{ad}_l$. This is clearly a state on $K(\mathcal{H})$, and we have

$$\begin{aligned}(\mathrm{id} \otimes \theta)\mathsf{ad}_l(T) &= (\mathrm{id} \otimes \phi \otimes \omega)(\mathrm{id} \otimes \mathsf{ad}_l)\mathsf{ad}_l(T) \\ &= (\mathrm{id} \otimes \phi \otimes \omega)(\Delta \otimes \mathrm{id})\mathsf{ad}_l(T) \\ &= 1 \otimes (\phi \otimes \omega)\mathsf{ad}_l(T) \\ &= 1\theta(T)\end{aligned}$$

for all $T \in K(\mathcal{H}_\pi)$, due to invariance of the Haar state ϕ. Hence θ is invariant.

Since $\pi = (\mathcal{H}, U)$ is irreducible we have $\mathrm{Mor}(\pi, \pi) = \mathbb{C}\,\mathrm{id}$ by Lemma 5.5, and therefore the only elements $X \in K(\mathcal{H})$ with $\mathsf{ad}_l(X) = 1 \otimes X$ are multiples of the identity. If we consider $T \in B(\mathcal{H})$ and set $\rho(T) = (\phi \otimes \mathrm{id})\mathsf{ad}_l(T)$, then

$$\begin{aligned}\mathsf{ad}_l(\rho(T)) &= (\phi \otimes \mathrm{id} \otimes \mathrm{id})(\mathrm{id} \otimes \mathsf{ad}_l)\mathsf{ad}_l(T) \\ &= (\phi \otimes \mathrm{id} \otimes \mathrm{id})(\Delta \otimes \mathrm{id})\mathsf{ad}_l(T) \\ &= 1 \otimes (\phi \otimes \mathrm{id})\mathsf{ad}_l(T) \\ &= 1 \otimes \rho(T),\end{aligned}$$

so that we get $\rho(T) \in \mathbb{C}1 \subset B(\mathcal{H})$. It follows that we can view ρ as a linear functional on $B(\mathcal{H})$.

Now assume that η is an invariant state on $K(\mathcal{H})$. Then

$$\begin{aligned}\eta(T) = \eta(T)\phi(1) &= \phi((\mathrm{id} \otimes \eta)\mathsf{ad}_l(T)) \\ &= (\phi \otimes \eta)\mathsf{ad}_l(T) \\ &= (\phi \otimes \phi \otimes \eta)(\Delta \otimes \mathrm{id})\mathsf{ad}_l(T) \\ &= (\phi \otimes \phi \otimes \eta)(\mathrm{id} \otimes \mathsf{ad}_l)\mathsf{ad}_l(T) \\ &= (\phi \otimes \eta)\mathsf{ad}_l(T) \\ &= \eta((\phi \otimes \mathrm{id})\mathsf{ad}_l(T)) \\ &= \eta(1)\rho(T) = \rho(T),\end{aligned}$$

and hence $\eta = \rho$. Applying this argument to the invariant state θ from above we get $\theta = \rho$, and thus $\eta = \rho = \theta$. In particular, we see that our initial construction of

θ is independent of the choice of the state ω, and that is gives the unique invariant state on $K(\mathcal{H})$.

The tricky bit is to show that θ is faithful. By general facts about states on finite dimensional matrix algebras, there is a positive element $Q \in B(\mathcal{H}) = K(\mathcal{H})$ with $\mathrm{tr}(Q) = 1$ such that $\theta(T) = \mathrm{tr}(TQ)$. In order to show that θ is faithful we have to prove that Q is invertible.

Let us choose an orthonormal basis $e_1, \dots, e_n$ of $\mathcal{H}$ diagonalising Q. Without loss of generality, we shall arrange the basis vectors such that $Q_{ii} > 0$ for $1 \leq i \leq d$ and $Q_{ii} = 0$ for $d < i \leq n$. Then our task is to show that $d = n$.

From the above description of $\theta = \rho$ we get

$$\begin{aligned} Q_{ij} 1 = \theta(e_{ij}) &= (\phi \otimes \mathrm{id})\mathsf{ad}_l(e_{ij}) \\ &= \sum_{m,n,k,l} (\phi \otimes \mathrm{id})((U^*_{nm} \otimes e_{mn})(1 \otimes e_{ij})(U_{kl} \otimes e_{kl})) \\ &= \sum_{m,l} \phi(U^*_{im} U_{jl}) e_{ml}. \end{aligned}$$

This implies $\phi(U^*_{im} U_{jl}) = 0$ if $i \neq j$, or if $i = j$ and $m \neq l$, and that $\phi(U^*_{il} U_{il}) = Q_{ii}$ is independent of l. Using these relations we obtain

$$\begin{aligned} Q_{ii} &= Q_{ii} \sum_k U^*_{kj} U_{kj} \\ &= \sum_k \phi(U^*_{ik} U_{ik}) U^*_{kj} U_{kj} \\ &= \sum_{k,l} \phi(U^*_{ik} U_{il}) U^*_{kj} U_{lj} \\ &= \phi(U^*_{ij} U_{ij}) \\ &= \sum_{k,l} U^*_{ik} U_{il} \phi(U^*_{kj} U_{lj}) \\ &= \sum_k U^*_{ik} U_{ik} \phi(U^*_{kj} U_{kj}) = \sum_k Q_{kk} U^*_{ik} U_{ik}. \end{aligned}$$

If $Q_{ii} = 0$ we get $Q_{kk} U^*_{ik} U_{ik} = 0$ for all k since the elements $U^*_{ik} U_{ik}$ are positive. This means $U_{ik} = 0$ for $d + 1 \leq i \leq n$ and $1 \leq k \leq d$, and can be interpreted as saying that U has upper triangular form.

In the notation of Proposition 6.4 this implies $\langle e_i, \lambda(f) e_k \rangle = \phi(U_{ik} f) = 0$ for all $f \in C(G)$ provided $d + 1 \leq i \leq n$ and $1 \leq k \leq d$. If $d < n$ we thus obtain a contradiction to Proposition 6.4 since π is assumed to be irreducible. Hence $d = n$ as required. □

We note that faithfulness of the invariant state in Theorem 6.6 is straightforward if the Haar state ϕ of $C(G)$ is assumed to be faithful. Indeed, under this assumption one can simply use that ad_l is injective, combined with the fact that the (minimal) tensor product of faithful states is faithful. This means in particular that Proposition 6.4 is not needed in the proof of Theorem 6.6 in this case.

There is also a right handed version of the above discussion. More precisely, if $\pi = (\mathcal{H}, U)$ is a finite dimensional unitary representation we can use the canonical flip map to view U as an element of $K(\mathcal{H}) \otimes C(G)$. The *right adjoint action* on $K(\mathcal{H})$ is the unital $*$-homomorphism $\mathsf{ad}_r : K(\mathcal{H}) \to K(\mathcal{H}) \otimes C(G)$ given by

$\mathsf{ad}_r(T) = U(T \otimes 1)U^*$. Using Lemma 6.5 one checks

$$
\begin{aligned}
(\mathrm{id} \otimes \Delta)\mathsf{ad}_r(e_{ij}) &= \sum_{k,l,m,n} (\mathrm{id} \otimes \Delta)(e_{kl} \otimes U_{kl})(e_{ij} \otimes 1)(e_{nm} \otimes U^*_{mn}) \\
&= \sum_{k,m} e_{km} \otimes \Delta(U_{ki}U^*_{mj}) \\
&= \sum_{k,l,m,n} e_{kl} \otimes U_{km}U^*_{ln} \otimes U_{mi}U^*_{nj} \\
&= \sum_{m,n} (\mathsf{ad}_r \otimes \mathrm{id})(e_{mn} \otimes U_{mi}U^*_{nj}) \\
&= (\mathsf{ad}_r \otimes \mathrm{id})\mathsf{ad}_r(e_{ij})
\end{aligned}
$$

for all $1 \leq i, j \leq n = \dim(\mathcal{H})$, and we conclude $(\mathrm{id} \otimes \Delta)\mathsf{ad}_r = (\mathsf{ad}_r \otimes \mathrm{id})\mathsf{ad}_r$.

Exercise 14. *Let $\pi = (\mathcal{H}, U)$ be an irreducible unitary representation. Show that $K(\mathcal{H})$ admits a unique invariant state with respect to the right adjoint action, and that this state is faithful.*

If $\mathcal{H}$ is a Hilbert space we write $\overline{\mathcal{H}} = \mathcal{H}^*$ for the conjugate Hilbert space. Recall that $\overline{\mathcal{H}}$ has the same underlying additive structure, but the scalar multiplication is conjugate to the one in $\mathcal{H}$. More precisely, for $\overline{\xi} \in \overline{\mathcal{H}}$ we have

$$\overline{c \cdot \xi} = \overline{c} \cdot \overline{\xi}$$

for all $c \in \mathbb{C}$. Moreover, the inner product of $\overline{\mathcal{H}}$ is determined by

$$\langle \overline{\xi}, \overline{\eta} \rangle = \langle \eta, \xi \rangle$$

in terms of the inner product of $\mathcal{H}$. We obtain an antimultiplicative $*$-isomorphism $j : B(\mathcal{H}) \to B(\overline{\mathcal{H}})$ such that $j(T)(\overline{\xi}) = \overline{T^*(\xi)}$ for $\xi \in \mathcal{H}$.

Now let $U \in C(G) \otimes K(\mathcal{H})$ be a finite dimensional unitary representation of G, and set $\overline{U} = (\mathrm{id} \otimes j)(U^*) \in C(G) \otimes K(\overline{\mathcal{H}})$. If $e_1, \dots, e_n$ is an orthonormal basis of $\mathcal{H}$ then $\overline{e_1}, \dots, \overline{e_n}$ is an orthonormal basis of $\overline{\mathcal{H}}$ and we get

$$
\begin{aligned}
\sum_{i,j} \overline{U}_{ij} \otimes |\overline{e_i}\rangle\langle\overline{e_j}| &= \overline{U} \\
&= (\mathrm{id} \otimes j)(U^*) \\
&= \sum_{i,j} (\mathrm{id} \otimes j)((U_{ij})^* \otimes |e_j\rangle\langle e_i|) \\
&= \sum_{i,j} (U_{ij})^* \otimes |\overline{e_i}\rangle\langle\overline{e_j}|,
\end{aligned}
$$

noting that

$$j(|e_j\rangle\langle e_i|)(\overline{e_k}) = \overline{|e_i\rangle\langle e_j|(e_k)} = \overline{\langle e_j, e_k\rangle}\overline{e_i} = |\overline{e_i}\rangle\langle\overline{e_j}|(\overline{e_k})$$

for all $1 \leq i, j, k \leq n$. This gives

$$\overline{U}_{ij} = (U_{ij})^*$$

for all $1 \leq i, j \leq n$.

Proposition 6.7. *Let $\pi = (\mathcal{H}, U)$ be a finite dimensional unitary representation of a compact quantum group G. Then there exists an invertible positive element $F \in B(\mathcal{H}) \cong B(\overline{\mathcal{H}})$ such that*

$$U^c = (\mathrm{id} \otimes F)\overline{U}(\mathrm{id} \otimes F^{-1}) \in C(G) \otimes K(\overline{\mathcal{H}})$$

is a unitary representation of G on $\overline{\mathcal{H}}$. If π is irreducible then F is uniquely determined up to a positive scalar.

Proof. Due to Theorem 6.6, the invariant state θ on $K(\mathcal{H})$ with respect to the left adjoint action is given by $\theta(T) = \mathrm{tr}(TQ)$ for a positive invertible matrix Q.

Note that the calculation

$$\begin{aligned}
Q_{ii}\delta_{ij} &= 1\theta(e_{ij}) \\
&= (\mathrm{id}\otimes\theta)(\mathsf{ad}_l(e_{ij})) \\
&= \sum_{m,n,k,l} (\mathrm{id}\otimes\theta)((U^*_{nm}\otimes e_{mn})(1\otimes e_{ij})(U_{kl}\otimes e_{kl})) \\
&= \sum_{m,n,k,l} U^*_{nm}U_{kl}\theta(e_{mn}e_{ij}e_{kl}) \\
&= \sum_{m,l} U^*_{ir}U_{jl}\theta(e_{ml}) = \sum_m U^*_{im}U_{jm}Q_{mm}
\end{aligned}$$

from the proof of Theorem 6.6 can be interpreted as saying $\overline{U}QU^t = Q$. Since Q is invertible we conclude that $\overline{U} = (U^*_{ij})$ is right invertible.

In the same way we can consider the right adjoint action $\mathsf{ad}_r : K(\mathcal{H}) \to K(\mathcal{H})\otimes C(G)$. Let us write η for the unique invariant state on $K(\mathcal{H})$, and note that $\eta(T) = \mathrm{tr}(TP)$ for some positive invertible matrix P by Exercise 14. Picking an orthonormal basis $f_1, \ldots, f_n$ on which P is diagonal, and using the corresponding matrix units f_{ij} we calculate

$$\begin{aligned}
P_{ii}\delta_{ij} &= 1\eta(f_{ij}) \\
&= (\eta\otimes\mathrm{id})(\mathsf{ad}_r(f_{ij})) \\
&= \sum_{m,n,k,l} (\eta\otimes\mathrm{id})((f_{kl}\otimes U_{kl})(f_{ij}\otimes 1)(f_{mn}\otimes U^*_{nm})) \\
&= \sum_{m,n,k,l} U_{kl}U^*_{nm}\eta(f_{kl}f_{ij}f_{mn}) \\
&= \sum_{n,k} U_{ki}U^*_{nj}\theta(f_{kn}) = \sum_k U_{ki}U^*_{kj}P_{kk}
\end{aligned}$$

for all i, j. This shows $U^tP\overline{U} = P$, which means that the matrix $\overline{U}$ is left invertible.

Combining these facts, we conclude that $\overline{U}$ is invertible, with inverse given by $QU^tQ^{-1} = P^{-1}U^tP$. Set $F = P^{1/2}$ and let

$$U^c = F\overline{U}F^{-1}.$$

Then

$$(U^c)^*U^c = (F^{-1}U^tF)(F\overline{U}F^{-1}) = F^{-1}U^tP\overline{U}F^{-1} = F^{-1}PF^{-1} = \mathrm{id}$$

and

$$U^c(U^c)^* = (F\overline{U}F^{-1})(F^{-1}U^tF) = (F\overline{U}F^{-1})(F^{-1}U^tF) = FP^{-1}F = \mathrm{id},$$

which means that U^c is unitary. Moreover we get

$$\begin{aligned}
(\Delta\otimes\mathrm{id})((U^c)_{ij}) &= \Delta(F_{ii}U^*_{ij}F^{-1}_{jj}) \\
&= F_{ii}\Delta(U^*_{ij})F^{-1}_{jj} \\
&= \sum_k F_{ii}U^*_{ik}\otimes U^*_{kj}F^{-1}_{jj} \\
&= \sum_k F_{ii}U^*_{ik}F^{-1}_{kk}\otimes F_{kk}U^*_{kj}F^{-1}_{jj} \\
&= \sum_k (U^c)_{ik}\otimes(U^c)_{kj},
\end{aligned}$$

so that Lemma 6.5 shows that U^c is a unitary representation of G on $\overline{\mathcal{H}}$. We will denote this representation by π^c.

If $T \in \mathrm{Mor}(\pi, \pi)$ then $Fj(T)F^{-1} \in \mathrm{Mor}(\pi^c, \pi^c)$ since

$$\begin{aligned}(\mathrm{id} \otimes Fj(T)F^{-1})U^c &= (\mathrm{id} \otimes Fj(T)F^{-1})(\mathrm{id} \otimes F)\overline{U}(\mathrm{id} \otimes F^{-1}) \\ &= (\mathrm{id} \otimes Fj(T))(\mathrm{id} \otimes j)(U^*)(\mathrm{id} \otimes F^{-1}) \\ &= (\mathrm{id} \otimes F)(\mathrm{id} \otimes j)(U^*(1 \otimes T))(\mathrm{id} \otimes F^{-1}) \\ &= (\mathrm{id} \otimes F)(\mathrm{id} \otimes j)((1 \otimes T)U^*)(\mathrm{id} \otimes F^{-1}) \\ &= (\mathrm{id} \otimes F)(\mathrm{id} \otimes j)(U^*)(\mathrm{id} \otimes j(T)F^{-1}) \\ &= U^c(\mathrm{id} \otimes Fj(T)F^{-1}).\end{aligned}$$

In particular, if π is irreducible then the same is true for π^c.

Finally, assume that $V = E\overline{U}E^{-1}$ is also a unitary representations for some positive invertible operator E. Then $T = EF^{-1}$ is a nonzero intertwiner from U^c to V. If π is irreducible then, by the previous observation, this must be a multiple of the identity due to Lemma 5.5. □

We note that the proof of Proposition 6.7 shows also that, up to a positive scalar multiple, the positive matrix P defining the invariant state on $K(\mathcal{H})$ with respect to the right adjoint action agrees with the inverse Q^{-1} of the matrix defining the invariant state on $K(\mathcal{H})$ with respect to the left adjoint action. In particular, there exists a state which is invariant for both the left and the right adjoint actions iff P and Q are multiples of the identity.

The unitary representation $\pi^c = (\mathcal{H}_c, U^c)$ on $\mathcal{H}_c = \overline{\mathcal{H}}$ obtained from $\pi = (\mathcal{H}, U)$ in Proposition 6.7 is called the *conjugate representation* of π and denoted π^c. Note that, a priori, the conjugate representation is defined only up to equivalence since the operator F in Proposition 6.7 is not unique.

However, if π is irreducible then the uniqueness statement in Proposition 6.7 means that there is a unique F such that $\mathrm{tr}(F^2) = \mathrm{tr}(F^{-2})$. For a general finite dimensional representation we get a uniquely determined F by stipulating

$$\mathrm{tr}(TF^2) = \mathrm{tr}(TF^{-2})$$

for all $T \in \mathrm{Mor}(\pi, \pi)$. We then define the *quantum dimension* of π by

$$\dim_q(\pi) = \mathrm{tr}(F^2) = \mathrm{tr}(F^{-2}),$$

where tr is the natural trace on $B(\mathcal{H})$, satisfying $\mathrm{tr}(1) = \dim(\mathcal{H})$. In the sequel, we will always fix this choice of F, so that the conjugate representation is defined on the nose, and not only up to equivalence, by the construction in Proposition 6.7.

Note that $\dim_q(\pi)$ equals the classical dimension $\dim(\pi) = \dim(\mathcal{H}_\pi)$ of the Hilbert space $\mathcal{H}_\pi$ iff $F = \mathrm{id}$.

Exercise 15. *Show that* $(\pi \otimes \eta)^c \cong \eta^c \otimes \pi^c$ *and* $(\pi^c)^c \cong \pi$ *for all finite dimensional unitary representations of* G.

Let us next introduce *matrix coefficients*. In fact, we have already been working with matrix coefficients in our discussion above, but we now give a formal definition. If $\pi = (\mathcal{H}_\pi, U^\pi)$ is a unitary representation of G then we call the elements $U^\pi_{\xi,\eta} \in C(G)$ for $\xi, \eta \in \mathcal{H}_\pi$ given by

$$U^\pi_{\xi,\eta} = (\mathrm{id} \otimes \omega_{\xi,\eta})(U^\pi)$$

the matrix coefficients of π. Here $\omega_{\xi,\eta} : K(\mathcal{H}_\pi) \to \mathbb{C}$ is the vector functional given by $\omega_{\xi,\eta}(T) = \langle \xi, T(\eta) \rangle$.

Definition 6.8. *We denote by* $\mathrm{Pol}(G) \subset C(G)$ *the vector space consisting of all matrix coefficients of finite dimensional unitary representations of* G.

Our aim is to show that $\mathrm{Pol}(G)$ is a Hopf $*$-algebra in a natural way. This will require a few intermediate steps.

Proposition 6.9. *The vector space* $\mathrm{Pol}(G)$ *is dense in* $L^2(G)$.

Proof. According to Theorem 6.3 there exists a set I such that we can decompose

$$L^2(G) \cong \bigoplus_{i\in I} \mathcal{H}^i$$

as a direct sum of irreducible unitary representations. Hence the space of all matrix coefficients $(\mathrm{id} \otimes \omega_{\xi,\eta})(W)$ such that $\xi, \eta \in \mathcal{H}^i$ for some $i \in I$ is dense in $[(\mathrm{id} \otimes \omega_{\xi,\eta})(W), \xi, \eta \in L^2(G)]$. The latter is equal to $C(G)$ by Proposition 6.2. □

We write $\mathsf{Irr}(G)$ for the set of equivalence classes of irreducible representations of G. For the sake of convenience it will be useful to fix an orthonormal basis of the Hilbert space $\mathcal{H}_\pi$ for (a representatives of) $\pi \in \mathsf{Irr}(G)$ such that the matrices F^π obtained in Proposition 6.7 are diagonal. In the sequel we shall call such bases *diagonal*. We will also always assume the normalisation $\mathrm{tr}((F^\pi)^2) = \mathrm{tr}((F^\pi)^{-2})$.

With this in place let us now derive relations between matrix coefficients as follows.

Proposition 6.10 (Schur orthogonality relations)**.** *Let G be a compact quantum group and $\pi, \eta \in \mathsf{Irr}(G)$. Then, with respect to diagonal orthonormal bases of $\mathcal{H}^\pi$ and $\mathcal{H}^\eta$, we have*

$$\phi((U^\pi_{ij})^* U^\eta_{kl}) = \delta_{\pi\eta} \frac{\delta_{jl}\delta_{ik}(F^\pi)^{-2}_{ii}}{\dim_q(\pi)}$$

and

$$\phi(U^\pi_{ij}(U^\eta_{kl})^*) = \delta_{\pi\eta} \frac{\delta_{ik}\delta_{jl}(F^\pi)^{2}_{jj}}{\dim_q(\pi)}$$

for all $1 \leq i, j \leq \dim(\pi), 1 \leq k, l \leq \dim(\eta)$.

Proof. We write $\mathsf{ad}_l : K(\mathcal{H}_\pi) \to C(G) \otimes K(\mathcal{H}_\pi), \mathsf{ad}_l(T) = (U^\pi)^*(1 \otimes T)U^\pi$ for the left adjoint action, and denote by $\theta : K(\mathcal{H}_\pi) \to \mathbb{C}$ the invariant state from Theorem 6.6.

Inspecting the arguments given in the proof of Theorem 6.6, and keeping in mind the definition $\dim_q(\pi) = \mathrm{tr}((F^\pi)^2) = \mathrm{tr}((F^\pi)^{-2})$ of the quantum dimension, we observe that $\theta(X) = \dim_q(\pi)^{-1} \mathrm{tr}(X(F^\pi)^{-2})$. We thus compute

$$\begin{aligned}
\frac{1}{\dim_q(\pi)} \mathrm{tr}(e_{ik}(F^\pi)^{-2}) &= \theta(e_{ik}) \\
&= (\phi \otimes \mathrm{id})\mathsf{ad}_l(e_{ik}) \\
&= \sum_{m,n,j,l} (\phi \otimes \mathrm{id})((U^\pi_{mj})^* \otimes e_{jm})(1 \otimes e_{ik})(U^\pi_{nl} \otimes e_{nl}) \\
&= \sum_{j,l} \phi((U^\pi_{ij})^* U^\pi_{kl}) e_{jl}.
\end{aligned}$$

This gives

$$\phi((U^\pi_{ij})^* U^\pi_{kl}) = \frac{\delta_{jl}\delta_{ik}(F^\pi)^{-2}_{ii}}{\dim_q(\pi)},$$

which is the first of the required orthogonality relations in the case $\pi = \eta$.

If $\pi \neq \eta$ we can consider the representation $\mathcal{H}^\pi \oplus \mathcal{H}^\eta$ and the corresponding left adjoint action on $K(\mathcal{H}^\pi \oplus \mathcal{H}^\eta)$. Since $(\phi \otimes \mathrm{id})\mathsf{ad}_l(T)$ is an intertwiner from $\mathcal{H}^\pi$ to $\mathcal{H}^\eta$ whenever $T \in B(\mathcal{H}^\pi, \mathcal{H}^\eta)$ the same calculation as above combined with Lemma 5.5 shows $\phi((U^\pi_{ij})^* U^\pi_{kl}) = 0$ for all i, j, k, l.

The remaining set of orthogonality relations is obtained in a similar way by considering the right adjoint action $\mathsf{ad}_r : K(\mathcal{H}_\pi) \to K(\mathcal{H}_\pi) \otimes C(G)$ instead. □

As a consequence of Proposition 6.10 we obtain linear bases of the vector space $\mathrm{Pol}(G)$ as follows.

Corollary 6.11. *For $\pi \in \mathsf{Irr}(G)$ let U^π_{ij} be the matrix coefficients of π with respect to a diagonal orthonormal basis of $\mathcal{H}_\pi$. Then the family of these elements, ranging over all $\pi \in \mathsf{Irr}(G)$ and $1 \leq i, j \leq \dim(\pi)$, is a vector space basis of $\mathrm{Pol}(G)$.*

Assume that we have chosen diagonal orthonormal bases of the Hilbert spaces $\mathcal{H}_\pi$ for all $\pi \in \mathsf{Irr}(G)$. Due to Corollary 6.11 we can define linear maps $\epsilon : \mathrm{Pol}(G) \to \mathbb{C}$ and $S : \mathrm{Pol}(G) \to \mathrm{Pol}(G)$ by setting

$$\epsilon(U^\pi_{ij}) = \delta_{ij}, \qquad S(U^\pi_{ij}) = (U^\pi_{ji})^*$$

for $1 \leq i, j \leq \dim(\pi)$. Then we clearly have

$$(\epsilon \otimes \mathrm{id})\Delta(U^\pi_{ij}) = \sum_k \epsilon(U^\pi_{ik})U^\pi_{kj} = U^\pi_{ij} = \sum_k U^\pi_{ik}\epsilon(U^\pi_{kj}) = (\mathrm{id} \otimes \epsilon)\Delta(U^\pi_{ij})$$

for all $\pi \in \mathsf{Irr}(G), 1 \leq i, j \leq \dim(\pi)$, and $\epsilon(1) = 1$. Note that we get in fact $(\epsilon \otimes \mathrm{id})(U) = 1 \otimes \mathrm{id}$ for any finite dimensional unitary representation $(\mathcal{H}, U)$. In particular, the above definition of ϵ does not depend on the choice of basis, and we obtain

$$(\epsilon \otimes \mathrm{id})(U^{\pi \otimes \eta}) = 1 \otimes \mathrm{id} = (\epsilon \otimes \mathrm{id})(U^\pi)(\epsilon \otimes \mathrm{id})(U^\eta)$$

for all $\pi, \eta \in \mathsf{Irr}(G)$. This means $\epsilon(U^\pi_{ij}U^\eta_{kl}) = \delta_{ij}\delta_{kl} = \epsilon(U^\pi_{ij})\epsilon(U^\eta_{kl})$ for all $1 \leq i, j \leq \dim(\pi), 1 \leq k, l \leq \dim(\eta)$, so that ϵ is an algebra homomorphism. Also, we have

$$1 \otimes \mathrm{id} = (\epsilon \otimes \mathrm{id})(U^{\pi^c}) = (\epsilon \otimes \mathrm{id})(F\overline{U^\pi}F^{-1}) = F(\epsilon \otimes \mathrm{id})(\overline{U^\pi})F^{-1}.$$

This means $\epsilon((U^\pi_{ij})^*) = \delta_{ij} = \epsilon(U^\pi_{ij})$ for all $\pi \in \mathsf{Irr}(G)$ and $1 \leq i, j \leq n$. It follows that ϵ is a $*$-homomorphism.

Similarly, we have $(S \otimes \mathrm{id})(U) = U^*$, so that the definition of S is independent of the choice of bases. We calculate

$$\mu(S \otimes \mathrm{id})\Delta(U_{ij}) = \sum_k m(U^*_{ki} \otimes U_{kj}) = \sum_k U^*_{ki}U_{kj} = \delta_{ij} = \epsilon(U_{ij}),$$

and in the same way one verifies

$$\mu(\mathrm{id} \otimes S)\Delta(U_{ij}) = \sum_k m(U_{ik} \otimes U^*_j) = \sum_k U_{ik}U^*_{jk} = \delta_{ij} = \epsilon(U_{ij}).$$

Summarising this discussion, we have now completed the proof of the following key result.

Theorem 6.12. *Let G be a compact quantum group. The linear space $\mathrm{Pol}(G)$ of matrix coefficients of G is naturally a unital Hopf $*$-algebra.*

The Hopf $*$-algebra $\mathrm{Pol}(G)$ admits a left and right invariant Haar functional given by the restriction of ϕ to $\mathrm{Pol}(G) \subset C(G)$. By construction, this functional is a state on $\mathrm{Pol}(G)$. From the Schur orthogonality relations it follows that ϕ is faithful on $\mathrm{Pol}(G)$, despite the fact that it need not be faithful on $C(G)$.

Choose a diagonal orthonormal basis of $\mathcal{H}_\pi$ for each $\pi \in \mathsf{Irr}(G)$. The *Woronowicz characters* $(f_z)_{z\in\mathbb{C}}$ are the linear functionals $f_z : \mathrm{Pol}(G) \to \mathbb{C}$ defined by

$$f_z(U^\pi_{ij}) = (F^\pi)^{2z}_{ij} = \delta_{ij}(F^\pi_{ii})^{2z}.$$

In the same way as in the construction of the counit ϵ of $\mathrm{Pol}(G)$ one checks that f_z is well-defined, and independent of the choice of diagonal basis. Note that we have in fact $f_0 = \epsilon$.

Lemma 6.13. *The Woronowicz characters $(f_z)_{z\in\mathbb{C}}$ satisfy the following relations.*

a) The functionals f_z are characters, that is, each f_z is an algebra homomorphism from $\mathrm{Pol}(G)$ to $\mathbb{C}$.

b) We have $f_z(g^*) = \overline{f_{-\overline{z}}(g)}$ *for all* $z \in \mathbb{C}$ *and* $g \in \mathrm{Pol}(G)$.
c) We have $(f_w \otimes f_z)\Delta(g) = f_{w+z}(g)$ *for all* $w, z \in \mathbb{C}$ *and* $g \in \mathrm{Pol}(G)$.

Proof. *a*) This follows from the normalisation of the matrices F^π for $\pi \in \mathsf{Irr}(G)$. Indeed, we have $(f_z \otimes \mathrm{id})(U) = F^{2z}$ for any finite dimensional unitary representation $(\mathcal{H}, U)$ of G, where F is the uniquely determined matrix described in the discussion following Proposition 6.7.
b) For any $\pi \in \mathsf{Irr}(G)$ we have $F^{\pi^c} = (F^\pi)^{-1}$, compare Exercise 15, and hence

$$f_z((U^\pi_{ij})^*) = f_z((\overline{U^\pi})_{ij}) = f_z(((F^\pi)^{-1} U^{\pi^c} F^\pi)_{ij}) = \delta_{ij}(F^\pi)^{-2z}_{ij} = \overline{f_{-\overline{z}}(U^\pi_{ij})}$$

for all $z \in \mathbb{C}$. This yields the claim due to Corollary 6.11.
c) We calculate

$$(f_w \otimes f_z)\Delta(U^\pi_{ij}) = \sum_k f_w(U^\pi_{ik}) f_z(U^\pi_{kj}) = \delta_{ij}(F^\pi_{ii})^{2w}(F^\pi_{jj})^{2z} = f_{w+z}(U^\pi_{ij})$$

for all $\pi \in \mathsf{Irr}(G)$ and $w, z \in \mathbb{C}$. Using again Corollary 6.11 yields the claim. □

Let us define a family $(\sigma_z)_{z\in\mathbb{C}}$ of linear endomorphisms of $\mathrm{Pol}(G)$ by

$$\sigma_z(U^\pi_{kl}) = \sum_{m,n} f_{iz}(U^\pi_{km}) U^\pi_{mn} f_{iz}(U^\pi_{nl}) = (F^\pi)^{2iz}_{kk} U^\pi_{kl} (F^\pi)^{2iz}_{ll}$$

for all $\pi \in \mathsf{Irr}(G)$ and $1 \leq k, l \leq \dim(\pi)$.

Exercise 16. *Show that the maps* σ_t *form a one-parameter group* $(\sigma_t)_{t\in\mathbb{R}}$ *of* $*$*-automorphisms of* $\mathrm{Pol}(G)$*. That is, show that each* σ_t *is a* $*$*-automorphism and that* $\sigma_{r+s} = \sigma_r\sigma_s$ *for all* $r, s \in \mathbb{R}$.

Using the one-parameter group of $*$-automorphisms $(\sigma_t)_{t\in\mathbb{R}}$ we can now discuss the KMS-property of the Haar state for $\mathrm{Pol}(G)$.

Proposition 6.14. *The Haar state* ϕ *satisfies* $\phi(\sigma_t(g)) = \phi(g)$ *for all* $g \in \mathrm{Pol}(G)$*. Moreover, for all* $g, h \in \mathrm{Pol}(G)$ *there exists a bounded continuous function* f *on the strip* $\{z \in \mathbb{C} \mid 0 \leq \Im(z) \leq 1\}$*, analytic in the interior, such that*

$$\phi(\sigma_t(g)h) = f_t, \qquad \phi(h\sigma_t(g)) = f_{t+i}$$

for all $t \in \mathbb{R}$.

Proof. The first claim is an immediate consequence of the fact that $\phi(U^\pi_{ij}) = 0$ whenever $\pi \in \mathsf{Irr}(G)$ is not the trivial representation, which in turn follows from Proposition 6.10.

In order to prove the second claim it suffices to consider the case that $g = U^\pi_{kl}, h = (U^\eta_{mn})^*$ are matrix elements of irreducible representations. Using Proposition 6.10 and Lemma 6.13 we calculate

$$\begin{aligned}\phi(\sigma_t(U^\pi_{kl})(U^\eta_{mn})^*) &= (F^\pi)^{2it}_{kk}(F^\pi)^{2it}_{ll}\phi(U^\pi_{kl}(U^\eta_{mn})^*) \\ &= (F^\pi)^{2it}_{kk}(F^\pi)^{2it}_{ll}\delta_{\pi\eta}\delta_{km}\delta_{ln}(F^\pi)^2_{ll}\end{aligned}$$

and

$$\begin{aligned}\phi((U^\eta_{mn})^*\sigma_t(U^\pi_{kl})) &= (F^\pi)^{2it}_{kk}(F^\pi)^{2it}_{ll}\phi((U^\eta_{mn})^* U^\pi_{kl}) \\ &= (F^\pi)^{2it}_{kk}(F^\pi)^{2it}_{ll}\delta_{\pi\eta}\delta_{ln}\delta_{km}(F^\pi)^{-2}_{kk}.\end{aligned}$$

Hence setting

$$f(z) = (F^\pi)^{2iz}_{kk}(F^\pi)^{2iz}_{ll}\delta_{\pi\eta}\delta_{km}\delta_{ln}(F^\pi)^2_{ll}$$

yields the claim. Explicitly, we have $f(t) = \phi(\sigma_t(g)h)$ for all $t \in \mathbb{R}$ by construction, and we also get

$$\begin{aligned} f(t+i) &= (F^\pi)^{2it}_{kk}(F^\pi)^{-2}_{kk}(F^\pi)^{2it}_{ll}(F^\pi)^{-2}_{ll}\delta_{\pi\eta}\delta_{km}\delta_{ln}(F^\pi)^2_{ll} \\ &= (F^\pi)^{2it}_{kk}(F^\pi)^{2it}_{ll}\delta_{\pi\eta}\delta_{ln}\delta_{km}(F^\pi)^{-2}_{kk} \\ &= \phi((U^\eta_{mn})^*\sigma_t(U^\pi_{kl})) = \phi(h\sigma_t(g)) \end{aligned}$$

as required. □

Note that the function f constructed in the proof of Proposition 6.14 is in fact analytic on all of $\mathbb{C}$.

It is a straightforward consequence of Proposition 6.10 that the restriction of the Haar state of $C(G)$ to $\mathrm{Pol}(G)$ is faithful. Using the KMS-property obtained in Proposition 6.14, one can show that ϕ is even faithful on the image $C^\mathsf{r}(G) = \widehat{\lambda}(C(G)) \subset B(L^2(G))$ of $C(G)$ under the GNS-representation. Explicitly, due to the invariance of ϕ under σ_t obtained in the first part of Proposition 6.14, the $*$-automorphisms σ_t induce unitary operators U_t on $L^2(G)$ by defining

$$U_t(\Lambda(g)) = \Lambda(\sigma_t(g))$$

for $g \in \mathrm{Pol}(G)$. This determines a unitary representation of $\mathbb{R}$ on $L^2(G)$, and the formula

$$\sigma_t(\widehat{\lambda}(g)) = U_t\widehat{\lambda}(g)U_t^*$$

yields a strongly continuous one-parameter group of $*$-automorphisms of $C^\mathsf{r}(G)$, extending the construction on the level of $\mathrm{Pol}(G)$. The second part of Proposition 6.14 then implies that ϕ is a KMS-state on $C^\mathsf{r}(G)$. Therefore ϕ is in fact faithful on $C^\mathsf{r}(G)$, see [3, Corollary 5.3.9].

Moreover, the coproduct of $C(G)$ induces a comultiplication on $C^\mathsf{r}(G)$. For this we can use the Kac-Takesaki operator W constructed at the beginning of this section, viewed as a unitary in $B(L^2(G) \otimes L^2(G))$. Then Proposition 6.2 means

$$W^*(1 \otimes \widehat{\lambda}(f))W = (\widehat{\lambda} \otimes \widehat{\lambda})\Delta(f)$$

for all $f \in C(G)$, which means that Δ determines a unital $*$-homomorphism $C^\mathsf{r}(G) \to C^\mathsf{r}(G) \otimes C^\mathsf{r}(G)$. Coassociativity and the density conditions for this map are immediate consequences of the corresponding properties for Δ, and we conclude that $C^\mathsf{r}(G)$ is a unital Hopf C^*-algebra.

This means that we have now come full circle. Keeping in mind that a weight on a unital C^*-algebra is densely defined iff it is a positive linear functional (defined everywhere), and that lower semicontinuity is automatic in this case, we see that the Haar state ϕ on $C^\mathsf{r}(G)$ is a lower semicontinuous, densely defined, faithful, left and right invariant KMS-weight. This means that $C^\mathsf{r}(G)$ is indeed the Hopf $*$-algebra underlying a compact quantum group G in the sense of Definition 3.7.

There is also a universal C^*-algebra $C^\mathsf{f}(G)$, obtained by taking the universal C^*-completion of $\mathrm{Pol}(G)$. It is not hard to check that the comultiplication of $\mathrm{Pol}(G)$ induces a unital $*$-homomorphism $C^\mathsf{f}(G) \to C^\mathsf{f}(G) \otimes C^\mathsf{f}(G)$ which turns $C^\mathsf{f}(G)$ into a Hopf C^*-algebra. The original Hopf C^*-algebra $C(G)$ is then an intermediate completion of $\mathrm{Pol}(G)$, and we get surjective morphisms

$$C^\mathsf{f}(G) \to C(G) \to C^\mathsf{r}(G)$$

of Hopf C^*-algebras. Roughly speaking, we may view all three Hopf C^*-algebras as different realisations of the same compact quantum group G.

To conclude this section let us record the *Peter-Weyl Theorem*, see [10, Theorem 5.12] for the classical version of this result.

Theorem 6.15 (Peter-Weyl Theorem)**.** *Let G be a compact quantum group. Then there is a unitary equivalence of unitary representations*

$$L^2(G) \cong \bigoplus_{\pi \in \mathsf{Irr}(G)} \mathcal{H}_\pi^{\oplus \dim(\mathcal{H}_\pi)},$$

decomposing the left regular representation of G.

Proof. It follows from Proposition 6.10 that the vectors

$$e_{ij}^\pi = \sqrt{\dim_q(\pi)} F_{ii}^\pi \Lambda(U_{ij}^\pi)$$

for $\pi \in \mathsf{Irr}(G), 1 \leq i, j \leq \dim(\pi)$ are mutually orthogonal in $L^2(G)$. Since $\mathrm{Pol}(G) \subset L^2(G)$ is dense by construction, they span in fact a dense linear subspace of $L^2(G)$ due to Proposition 6.9. Hence these vectors form an orthonormal basis of $L^2(G)$.

Using again Proposition 6.10 we obtain

$$\begin{aligned}
(\mathrm{id} \otimes \omega_{e_{kl}^\pi, e_{mn}^\eta})(W^*) &= \sqrt{\dim_q(\pi)}\sqrt{\dim_q(\eta)} F_{kk}^\pi F_{mm}^\eta (\mathrm{id} \otimes \omega_{\Lambda(U_{kl}^\pi), \Lambda(U_{mn}^\eta)})(W^*) \\
&= \sqrt{\dim_q(\pi)}\sqrt{\dim_q(\eta)} F_{kk}^\pi F_{mm}^\eta \sum_j U_{mj}^\eta \langle \Lambda(U_{kl}^\pi), \Lambda(U_{jn}^\eta)\rangle \\
&= \delta_{\pi\eta} \dim_q(\pi) F_{kk}^\pi F_{mm}^\pi U_{mk}^\pi \frac{1}{\dim_q(\pi)} (F^\pi)_{kk}^{-2} \delta_{ln} \\
&= \delta_{\pi\eta} (F^\pi)_{kk}^{-1} F_{mm}^\pi U_{mk}^\pi \delta_{ln},
\end{aligned}$$

or equivalently,

$$(\mathrm{id} \otimes \omega_{e_{mn}^\eta, e_{kl}^\pi})(W) = \delta_{\pi\eta} (F^\pi)_{kk}^{-1} F_{mm}^\pi (U_{mk}^\pi)^* \delta_{ln}.$$

Comparing this with

$$(U^{\pi^c})_{mk} = (F^\pi \overline{U^\pi} (F^\pi)^{-1})_{mk} = F_{mm}^\pi (U_{mk}^\pi)^* (F^\pi)_{kk}^{-1},$$

it follows that the regular representation leaves the linear span of the vectors e_{mn}^π for $1 \leq m, n \leq \dim(\mathcal{H}_\pi)$ invariant, and decomposes this vector space into $\dim(\mathcal{H}_\pi)$ copies of the representation π^c. This yields the claim. □

7. Actions

In group theory it is natural and important to consider actions on various types of objects. For quantum groups the situation is no different.

Let us introduce actions of quantum groups on C^*-algebras. In fact, the following definition makes sense, with obvious modifications, for general Hopf C^*-algebras.

Definition 7.1. *A (continuous, left) action of a locally compact quantum group G on a C^*-algebra A is an injective nondegenerate $*$-homomorphism $\alpha : A \to M(C_0^{\mathsf{r}}(G) \otimes A)$ such that the diagram*

$$\begin{array}{ccc}
A & \xrightarrow{\quad\alpha\quad} & M(C_0^{\mathsf{r}}(G) \otimes A) \\
\Big\downarrow{\scriptstyle\alpha} & & \Big\downarrow{\scriptstyle\Delta \otimes \mathrm{id}} \\
M(C_0^{\mathsf{r}}(G) \otimes A) & \xrightarrow{\mathrm{id} \otimes \alpha} & M(C_0^{\mathsf{r}}(G) \otimes C_0^{\mathsf{r}}(G) \otimes A)
\end{array}$$

is commutative and $[\alpha(A)(C_0^{\mathsf{r}}(G) \otimes 1)] = C_0^{\mathsf{r}}(G) \otimes A$.
A G-C^*-algebra (A, α) is a C^*-algebra A with an action α of G on A. If (A, α) and (B, β) are G-C^*-algebras, then a $*$-homomorphism $f : A \to M(B)$ is called G-equivariant if $\beta f = (\mathrm{id} \otimes f)\alpha$.

We write G-Alg for the category whose objects are G-C^*-algebras and morphisms given by equivariant $*$-homomorphisms.

Example 7.2. *Let us consider two basic examples of G-C^*-algebras.*

a) If $A = C_0^{\mathrm{r}}(G)$ then $\alpha = \Delta$ defines an action of G on A which makes A a G-C^*-algebra.
b) For any C^*-algebra B the trivial action $\beta : B \to M(C_0^{\mathrm{r}}(G) \otimes B), \beta(b) = 1 \otimes b$ turns B into an G-C^*-algebra.

The actions in Example 7.2 do not rely on any special properties of the quantum groups under consideration. For more interesting examples we need to invoke the specific structure at hand in different situations. A well-studied class of actions arises from homogeneous spaces for $SU_q(2)$, see [24].

Example 7.3 (Podleś sphere)**.** *Consider $G = SU_q(2)$. Then the classical torus $T = S^1$ is a closed quantum subgroup of $SU_q(2)$. By definition, this means that there is a morphism of Hopf C^*-algebras $C(SU_q(2)) \to C(T)$ determined by*

$$\pi \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix} = \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}$$

on generators. Moreover, the formula $\rho = (\mathrm{id} \otimes \pi)\Delta : C(G) \to C(G) \to C(T)$ defines a (right) action of T on $C(G) = C(SU_q(2))$. The (standard) Podleś sphere $SU_q(2)/T$ is the space of coinvariants with respect to this action, that is,

$$C(SU_q(2)/T) = \{f \in C(SU_q(2)) | (\mathrm{id} \otimes \pi)\Delta(f) = f \otimes 1\},$$

We claim that the formula $\alpha = \Delta$ defines an action $\alpha : C(G/T) \to C(G) \otimes C(G/T)$. To see that α is a well-defined $*$-homomorphism we observe that $f \in C(SU_q(2)/T)$ iff $(\mathrm{id} \otimes \phi)\rho(f) = f$, where ϕ is the Haar state of $C(T)$. Combining this with coassociativity of Δ it follows that α maps $C(G/T)$ indeed into $C(G) \otimes C(G/T)$, and it is obvious that we have $(\Delta \otimes \mathrm{id})\alpha = (\mathrm{id} \otimes \alpha)\alpha$. The density condition for α is an easy consequence of the density condition $[\Delta(C(G))(C(G) \otimes 1)] = C(G) \otimes C(G)$ for the quantum group, using again averaging with respect to the Haar state of $C(T)$.

In the classical case $q = 1$, the C^*-algebra $C(SU_1(2))$ identifies canonically with the algebra of continuous functions on the 2-sphere $SU(2)/T = S^2$.

Let us also consider the action which is the starting point for the definition of the quantum permutation group S_n^+.

Lemma 7.4. *The quantum permutation group S_n^+ acts on the commutative C^*-algebra $\mathbb{C}^n$ via*

$$\alpha(p_i) = \sum_j u_{ij} \otimes p_j$$

where $p_1, \dots, p_n$ are the canonical minimal projections in $\mathbb{C}^n$.

Proof. First we need to verify that the above formula defines a unital $*$-homomorphism $\mathbb{C}^n \to C(S_n^+) \otimes \mathbb{C}^n$. To this end note that $\alpha(p_i)$ is a projection since each u_{ij} is a projection, and that $\alpha(p_i)\alpha(p_j) = 0$ for $i \neq j$. Moreover we have

$$\alpha(1) = \sum_i \alpha(p_i) = \sum_{i,j} u_{ij} \otimes p_j = \sum_j 1 \otimes p_j = 1 \otimes 1$$

by the defining relations of $C(S_n^+)$.

The coaction identity $(\mathrm{id} \otimes \alpha)\alpha = (\Delta \otimes \mathrm{id})\alpha$ is obvious. To conclude the proof we need to check the density condition. For $f \in C(S_n^+)$ and $1 \leq k \leq n$ we have

$$\sum_i (f u_{ik} \otimes 1)\alpha(p_i) = \sum_{i,j} f u_{ik} u_{ij} \otimes p_j = \sum_j \delta_{jk} f \otimes p_k = f \otimes p_k.$$

Since elements of this form linearly span $C(S_n^+) \otimes \mathbb{C}^n$ the claim follows. □

Exercise 17. *Assume that G is a compact quantum group acting on the C^*-algebra $\mathbb{C}^n$, via $\beta : \mathbb{C}^n \to C(G) \otimes \mathbb{C}^n$. Show that there is a unique morphism of Hopf C^*-algebras $f : C(S_n^+) \to C(G)$ such that $(f \otimes \mathrm{id})\alpha = \beta$.*

Exercise 17 can be viewed as saying that S_n^+ is the *universal compact quantum group acting on n points*. This is an analogue, in the world of compact quantum groups, of the fact that any action of a (compact) group on a set with n elements factorises (uniquely) through the symmetric group S_n.

Let us come back to the example of the adjoint action which was introduced in Section 6.

Exercise 18. *Let G be a compact quantum group and let $\pi = (\mathcal{H}, U)$ be a finite dimensional unitary representation of G. Show that the left adjoint action of G on $K(\mathcal{H})$, given by $\mathsf{ad}_l(T) = U^*(1 \otimes T)U$, turns $K(\mathcal{H})$ into a G-C^*-algebra.*

8. The noncommutative geometry of the Podleś sphere

In this section we discuss some constructions and results related to $SU_q(2)$ and the standard Podleś sphere. Background material on compact quantum groups and q-deformations can be found in [11].

Let us first recall the definition of the quantum group $SU_q(2)$. In Definition 4.3 we introduced the C^*-algebra $C(SU_q(2))$ as the universal C^*-algebra generated by elements α and γ satisfying the relations saying that the fundamental matrix

$$u = \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix}$$

is unitary. Explicitly, this is equivalent to the relations

$$\alpha\gamma = q\gamma\alpha, \quad \alpha\gamma^* = q\gamma^*\alpha, \quad \gamma\gamma^* = \gamma^*\gamma, \quad \alpha^*\alpha + \gamma^*\gamma = 1, \quad \alpha\alpha^* + q^2\gamma\gamma^* = 1.$$

Recall also the comultiplication $\Delta : C(SU_q(2)) \to C(SU_q(2)) \otimes C(SU_q(2))$ given by

$$\Delta(\alpha) = \alpha \otimes \alpha - q\gamma^* \otimes \gamma, \qquad \Delta(\gamma) = \gamma \otimes \alpha + \alpha^* \otimes \gamma,$$

or equivalently,

$$\Delta \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix} = \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix} \otimes \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix}$$

in matrix notation.

The $*$-algebra $\mathrm{Pol}(SU_q(2))$ of matrix coefficients for $SU_q(2)$ is the $*$-subalgebra of $C(SU_q(2))$ generated by α and γ. It is a Hopf-$*$-algebra with comultiplication $\Delta : \mathrm{Pol}(SU_q(2)) \to \mathrm{Pol}(SU_q(2)) \otimes \mathrm{Pol}(SU_q(2))$ given by the same formula as above. Using again matrix notation, the counit $\epsilon : \mathrm{Pol}(SU_q(2)) \to \mathbb{C}$ is defined by

$$\epsilon \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and the antipode $S : \mathrm{Pol}(SU_q(2)) \to \mathrm{Pol}(SU_q(2))$ by

$$S \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix} = \begin{pmatrix} \alpha^* & \gamma^* \\ -q\gamma & \alpha \end{pmatrix}.$$

We write $L^2(SU_q(2))$ for the Hilbert space obtained as the completion of $C(SU_q(2))$ with respect to the inner product

$$\langle \Lambda(f), \Lambda(g) \rangle = \phi(f^*g)$$

for $f, g \in C(SU_q(2))$, where ϕ is the Haar state. It is equipped with the left regular representation of $SU_q(2)$, given by the Kac-Takesaki operator.

The representation theory of $SU_q(2)$ is very similar to its classical counterpart. More precisely, the set $\mathsf{Irr}(SU_q(2))$ of equivalence classes of irreducible representations can be identified with $\frac{1}{2}\mathbb{N}_0$, and the irreducible representation $V(l)$ corresponding to "spin" $l \in \frac{1}{2}\mathbb{N}_0$ has dimension $2l+1$.

According to the Peter-Weyl theorem 6.15, the Hilbert space $L^2(SU_q(2))$ has an orthonormal basis given by the decomposition of the left regular representation into the isotypical components corresponding to $l \in \frac{1}{2}\mathbb{N}_0$. Explicitly, we obtain an orthonormal basis consisting of vectors $e^{(l)}_{i,j}$ where $l \in \frac{1}{2}\mathbb{N}$ and $-l \leq i, j \leq l$ for all l, by suitably normalising the matrix coefficients of $V(l)$.

Recall also the definition of the Podleś sphere $SU_q(2)/T$ from Section 7. Inside $C(SU_q(2)/T)$ we have the dense $*$-subalgebra

$$\mathrm{Pol}(SU_q(2)/T) = \{f \in \mathrm{Pol}(SU_q(2)) | (\mathrm{id} \otimes \pi)\Delta(f) = f \otimes 1\}$$

corresponding to polynomial functions. Generalising this construction, we define for $k \in \mathbb{Z}$ the space

$$\Gamma(E_k) = \{x \in \mathrm{Pol}(SU_q(2)) | (\mathrm{id} \otimes \pi)\Delta(x) = x \otimes z^k\} \subset \mathrm{Pol}(SU_q(2)),$$

and we let $C(E_k)$ and $L^2(E_k)$ be the closures of $\Gamma(E_k)$ in $C(SU_q(2))$ and $L^2(SU_q(2))$, respectively. The space $\Gamma(E_k)$ is a $\mathrm{Pol}(SU_q(2)/T)$-bimodule in a natural way. It can be shown that $\Gamma(E_k)$ is finitely generated and projective both as a left and right $\mathrm{Pol}(SU_q(2)/T)$-module. This follows from the fact that $\mathrm{Pol}(SU_q(2)/T) \subset \mathrm{Pol}(SU_q(2))$ is a faithfully flat Hopf-Galois extension, see [26]. Similarly, the vector space $C(E_k)$ is naturally a $SU_q(2)$-equivariant Hilbert $C(SU_q(2)/T)$-module. The space $L^2(E_k)$ is naturally a representation of $SU_q(2)$. These structures are induced from $C(SU_q(2))$ and $L^2(SU_q(2))$ by restriction.

In the classical case $q = 1$, the above constructions correspond to looking at induced vector bundles over the homogeneous space $SU(2)/T \cong S^2$. More precisely, if $\mathbb{C}_k$ is the irreducible representation of T of weight $k \in \mathbb{Z}$, then

$$\Gamma(E_k) = \Gamma(SU(2) \times_T \mathbb{C}_k)$$

is the space of polynomial sections of the vector bundle

$$SU(2) \times_T \mathbb{C}_k = (SU(2) \times \mathbb{C})/\sim$$

over $SU(2)/T$ where

$$(gt, \lambda) \sim (g, t^k \lambda)$$

for all $g \in SU(2), \lambda \in \mathbb{C}$ and $t \in T$. Similarly, $C(E_k)$ and $L^2(E_k)$ are the spaces of continuous sections and L^2-sections, respectively, in this case.

We now recall the definition of the equivariant spectral triple for the Podleś sphere due to Dąbrowski and Sitarz [8]. The underlying graded representation of $SU_q(2)$ is

$$\mathcal{H} = L^2(E_1) \oplus L^2(E_{-1})$$

as defined above. The representation of $\mathcal{A} = \mathrm{Pol}(SU_q(2)/T)$ is given by left multiplication. Moreover one obtains a G-equivariant self-adjoint unbounded operator D on $\mathcal{H}$ by

$$D = \begin{pmatrix} 0 & E \\ F & 0 \end{pmatrix},$$

where $E, F \in U_q(\mathfrak{sl}(2, \mathbb{C}))$ are the generators of the quantized universal enveloping algebra associated with the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$. The spectral triple $(\mathcal{A}, \mathcal{H}, D)$ has a real structure, and a suitable version of the local index formula applies, see [8], [22] for more information.

It follows from the Peter-Weyl Theorem 6.15 for $SU_q(2)$ that the underlying $SU_q(2)$-representations of $L^2(E_1)$ and $L^2(E_{-1})$ are equivalent. In particular, taking the bounded transform of the operator D yields G-equivariant self-adjoint unitary operator F on $\mathcal{H}$ such that

$$F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

by identifying the basis vectors $e^{(l)}_{i,1/2}$ and $e^{(l)}_{i,-1/2}$ of the irreducible components in even and odd degrees.

Proposition 8.1. *With the notation as above, $D = (\mathcal{H}, \phi, F)$ is a $SU_q(2)$-equivariant Fredholm module defining an element in $KK^{SU_q(2)}(C(SU_q(2)/T), \mathbb{C})$.*

Classically, we can view $SU(2)/T \cong SL(2, \mathbb{C})/B$ where $B \subset SL(2, \mathbb{C})$ is the Borel subgroup of upper triangular matrices. It turns out that $SL(2, \mathbb{C})$ and B admit natural deformations $SL_q(2, \mathbb{C})$ and B_q, respectively. For $SL(2, \mathbb{C})$ this is given by the Drinfeld double $\mathsf{D}(SU_q(2))$ of $SU_q(2)$, which is defined in a similar way as the construction we considered for finite quantum groups in section 2.

Moreover, we have $C(SU_q(2)/T) \cong C(SL_q(2, \mathbb{C})/B_q)$ canonically. The key point here is that this allows one to view the Podleś sphere as a homogeneous space for the noncompact quantum group $SL_q(2, \mathbb{C})$, and in particular as a G-C^*-algebra for $SL_q(2, \mathbb{C})$.

Theorem 8.2. *Let $q \in (0, 1]$. The Fredholm module defined above induces an element $[D]$ in $KK^{\mathsf{D}(SU_q(2))}(C(SU_q(2)/T), \mathbb{C})$. Moreover the standard Podleś sphere $C(SU_q(2)/T)$ is equivalent to $\mathbb{C} \oplus \mathbb{C}$ in $KK^{\mathsf{D}(SU_q(2))}$.*

For a proof of Theorem 8.2 we refer to [29]. This result about the Podleś sphere is the key ingredient in the proof of the Baum-Connes conjecture for free orthogonal quantum groups in [29].

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